Topping Up and Optimal Subsidies

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Motivation

Governments **redistribute** through subsidies for specific goods and services:

- **food assistance** (e.g., SNAP);
- **housing** (e.g., public housing programs, LIHTC);
- other examples: child care, education, transportation, health care, energy.

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Topping up can also be a **design choice** of subsidy programs:

- Implementation through consumer vouchers or institutional subsidies?
- Explicit banning? e.g. housing loan subsidies that depend on apartment size.

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> *"Programs with or without the possibility of topping up have different welfare properties. Currently, there are no general results regarding the merits of the two. [...] Nor are there any general results on the characterization of optimal public provision policies, targeted or universal."*
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Approach: Study a nonlinear in-kind subsidy for a good also supplied by a private market.

- Types differ in demand θ and welfare weight $\omega(\theta)$.
- The planner faces opportunity cost α of public funds.
- Cash transfers are restricted, so redistribution works occurs solely via the subsidy schedule.

This Paper

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Main results:

- characterize when topping up affects the scope for subsidies; and
- characterize how topping up affects the optimal design of subsidies.

Related Work

- **Redistributive Mechanism Design.** Weitzman (1977), Che, Gale and Kim (2013), Condorelli (2013), Dworczak, Kominers and Akbarpour (2021), Akbarpour, Dworczak and Kominers (2022, 2024), Kang (2023, 2024), Akbarpour, Budish, Dworczak and Akbarpour (2024), Pai and Strack (2024).
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- **"Partial" Mechanism Design.** Philippon and Skreta (2012), Tirole (2012), Fuchs and Skrzypacz (2015), Dworczak (2020), Loertscher and Muir (2022), Loertscher and Marx (2022), Kang and Muir (2022), Kang (2023).
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- **Methodological Tools in Mechanism Design.**
  - Generalized ironing: Toikka (2011).
  - Lagrangian approach: Amador, Werning and Angeletos (2006), Amador and Bagwell (2013).
  - Type-dependent outside options: Jullien (2000), Dworczak and Muir (2024), Corrao, Flynn and Sastry (2023),Valenzuela-Stookey and Poggi (2024).
  - Majorization and convex-order tools: Kleiner, Moldovanu and Strack (2021, 2026), Yang and Zentefis (2024).
  - ⇝ **This paper:** majorization constraints due to type-dependent outside options in a convex program.

Related Work

- **Public Finance.**

  Theory: Ramsey (1927), Diamond (1975), Mirrlees (1976, 1986), Atkinson and Stiglitz (1976), Nichols and Zeckhauser (1982), Hammond (1987), Blackorby and Donaldson (1988), Coate (1989), Besley and Coate (1991), Coate, Johnson and Zeckhauser (1994), Blomquist and Christiansen (1998), Gahvari and Mattos (2007).

  Empirics: Baum-Snow and Marion (2009), Diamond and McQuade (2019), van Dijk (2019), Dinerstein, Neilson and Otero (2020), Atal, Cuesta, Gonzalez and Otero (2021), Dinerstein and Smith (2021), Jimenez-Hernandez and Seira (2021), Handbury and Moshary (2021).

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  - Atkinson and Stiglitz (1976) theorem fails with preference heterogeneity: Saez (2002), Pai and Strack (2024).
  - Related recent theory: Doligalski, Dworczak, Akbarpour and Kominers (2025), Doligalski, Dworczak, Krysta and Tokarski (2025).

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  As Doligalski, Dworczak, Akbarpour and Kominers (2025) write:

  "Several decades after the original work of Atkinson and Stiglitz (1976), conventional economic wisdom seems to have embraced the idea of using income taxes rather than goods market interventions to redistribute; it would appear that this intuition needs to be revisited, and more research is needed to understand whether it constitutes good policy advice under realistic scenarios."

Model

Setup

Consumers:

- There is a unit mass of risk-neutral consumers in market for a divisible, homogeneous good.
- Consumers differ in type $\theta \in [\underline{\theta}, \overline{\theta}]$ with $\underline{\theta} \geq 0$, and $\theta \sim F$, continuous with density $f > 0$.
- Each consumer derives utility $\theta v(q) - t$ from quantity/quality $q \in [0, A]$ given payment t.
  $v : [0, A] \to \mathbb{R}$ is differentiable with $v' > 0$, $v'' < 0$ and $v' \to \infty$ as $q \downarrow 0$.

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Extensions (not today): equilibrium effects, observable characteristics, product choice and eligibility.

Model Overview

Laissez-Faire Equilibrium

- Perfectly competitive private market $\rightsquigarrow$ **laissez-faire price** $p^{\text{LF}} = c$ per unit.
- Each consumer solves

$$U^{\text{LF}}(\theta) := \max_{q \in [0,A]} \left[\theta v(q) - cq\right].$$

v is strictly concave $\rightsquigarrow$ unique maximizer:

$$q^{\text{LF}}(\theta) = (v')^{-1}\left(\frac{c}{\theta}\right) = D(c, \theta).$$

- To simplify statements of some results, assume today that $q^{\text{LF}}(\underline{\theta}) > 0$.

▸ Model Overview

Subsidy Design

Social planner costlessly contracts with firms and sells units at a **subsidized payment schedule** $P^{\sigma}(q)$.

$\rightsquigarrow \Sigma(q) = cq - P^{\sigma}(q)$ is the **total subsidy** as a function of q, and $\sigma(q) = \Sigma'(q)$ is the **marginal subsidy**.

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Key assumption: The social planner can subsidize consumption but **not make lump-sum cash transfers**,

$$\rightsquigarrow P^{\sigma}(q) \geq 0 \text{ for all } q.$$

▸ Taxation

Private Market Interaction

Topping Up: given any price schedule $P(q)$, the effective price schedule is the c-Lipschitz minorant $P^{\text{eff}}(q) = \min_{q' \leq q}\{P(q') + c(q - q')\}$

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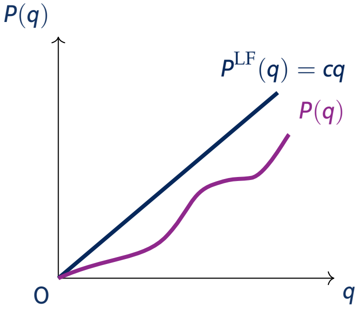

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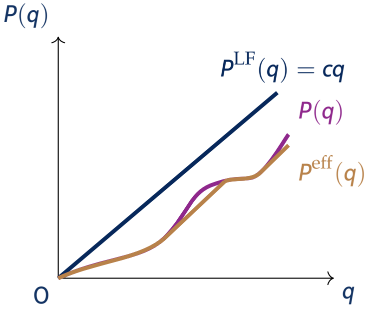

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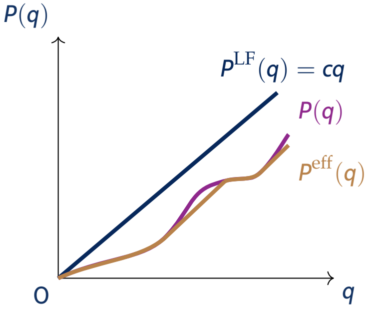

Direct mechanism: $P'(q) \le c$

$\iff q(\theta) \ge q^{\mathrm{LF}}(\theta)$

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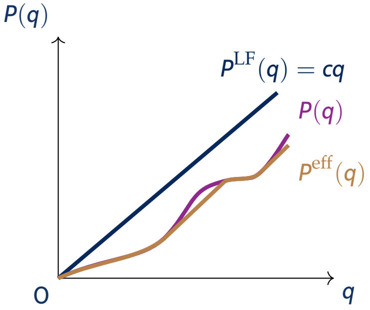

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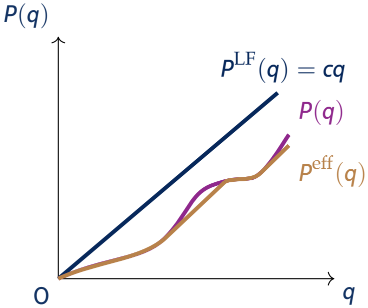

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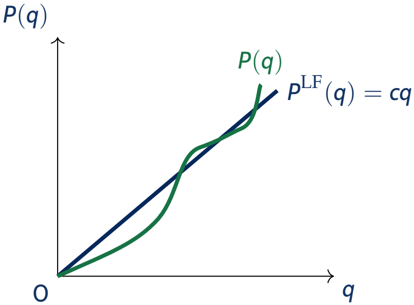

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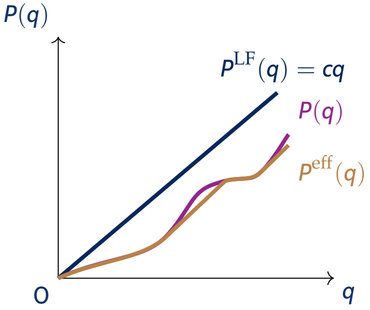

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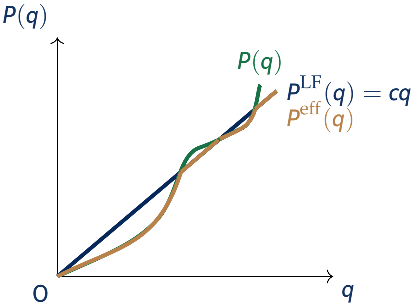

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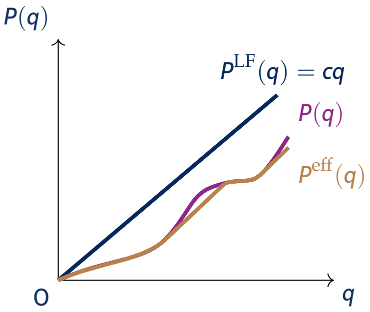

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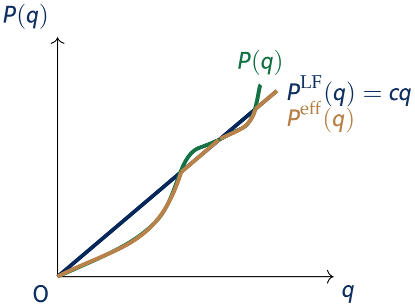

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$\iff U(\theta) \geq U^{\mathrm{LF}}(\theta)$

$\iff \Sigma(q) \geq 0$.

Redistributive Objective

The social planner seeks to maximize **weighted total surplus**.

- **Consumer surplus**: social planner assigns a welfare weight $\omega(\theta) := \mathbf{E}[\omega|\theta]$ to consumer type θ.
  - $\rightsquigarrow$ $\omega(\theta)$: expected social value of giving consumer θ one unit of money.

▸ Endogenizing ω and α

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⇝ **Objective**:

$$\max_{P^\sigma(q)\geq 0} \int_\theta \left[\omega(\theta)U^\sigma(\theta) - \alpha\Sigma(q^\sigma(\theta))\right] \mathrm{d}F(\theta) \text{ s.t. } \underbrace{\sigma(q) \geq 0}_{\text{topping up}} \text{ or } \underbrace{\Sigma(q) \geq 0}_{\text{no topping up}}$$

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Remarks:

- If $\omega(\theta) > \alpha$, social planner would want to transfer a dollar to type θ.
- If $\mathbf{E}_\theta[\omega(\theta)] > \alpha$, social planner would want to make a lump-sum cash transfer to all consumers.

▸ Endogenizing ω and α

Mechanism Design Problem

The social planner chooses **total allocation function** q and **total payment function** t to maximize weighted total surplus:

$$\max_{(q,t)} \int_{\underline{\theta}}^{\overline{\theta}} \Big[\omega(\theta) \underbrace{[\theta v(q(\theta)) - t(\theta)]}_{\text{consumer surplus}} - \alpha \underbrace{[cq(\theta) - t(\theta)]}_{\text{net cost}}\Big] \, dF(\theta),$$

subject to

- incentive compatibility, $$\theta \in \arg\max_{\hat{\theta} \in [\underline{\theta}, \overline{\theta}]} \left[\theta v(q(\hat{\theta})) - t(\hat{\theta})\right] \quad \forall\, \theta \in [\underline{\theta}, \overline{\theta}]; \qquad \text{(IC)}$$
- no lump-sum transfers, $$t(\theta) \geq 0 \quad \forall\, \theta \in [\underline{\theta}, \overline{\theta}]; \qquad \text{(LS)}$$
- individual rationality, $$\theta v(q(\theta)) - t(\theta) \geq U^{\text{LF}}(\theta) \quad \forall\, \theta \in [\underline{\theta}, \overline{\theta}], \qquad \text{(IR)}$$
- topping up constraint, $$q(\theta) \geq q^{\text{LF}}(\theta) \quad \forall \theta \in [\underline{\theta}, \overline{\theta}]. \qquad \text{(TU)}$$

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subject to (IC), (LS), (IR), and (TU).

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#1. Apply Myerson (1981) Lemma and Milgrom and Segal (2002) envelope theorem to express objective in terms of $\underline{U} := U(\underline{\theta})$ and $q(\theta)$ non-decreasing, substituting

$$t(\theta) = \theta v(q(\theta)) - \int_{\underline{\theta}}^{\theta} v(q(s))\, \mathrm{d}s - \underline{U}.$$

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$$\max_{\underline{U},\, q \text{ non-decreasing}} \left\{ \left[\mathbf{E}[\omega] - \alpha\right] \underline{U} + \int_{\underline{\theta}}^{\overline{\theta}} \left[\left[\alpha\theta + \frac{\int_{\theta}^{\overline{\theta}} \left[\omega(s) - \alpha\right] \, \mathrm{d}F(s)}{f(\theta)} \right] v(q(\theta)) - \alpha c q(\theta) \right] \, \mathrm{d}F(\theta) \right\},$$

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subject to (LS), (IR), and (TU).

#2. Suffices to enforce (LS) only for lowest type $\underline{\theta}$ because $t(\theta)$ is nondecreasing by (IC), so

$$\underline{U} \leq \underline{\theta} v(q(\underline{\theta})),$$

while (IR) for $\underline{\theta}$ implies

$$\underline{U} \geq U^{\mathrm{LF}}(\underline{\theta}).$$

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The social planner chooses **total allocation function** q to maximize weighted total surplus:

$$\max_{\substack{U^{LF}(\underline{\theta}) \leq \underline{U} \leq \underline{\theta} v(q(\underline{\theta})), \\ q \text{ non-decreasing}}} \left\{ [\mathbf{E}[\omega] - \alpha]\, \underline{U} + \int_{\underline{\theta}}^{\overline{\theta}} \left[\left[\alpha\theta + \frac{\int_{\theta}^{\overline{\theta}} [\omega(s) - \alpha]\, \mathrm{d}F(s)}{f(\theta)} \right] v(q(\theta)) - \alpha c q(\theta) \right] \mathrm{d}F(\theta) \right\},$$

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#3. Writing virtual type

$$J(\theta) = \underbrace{\theta}_{\text{efficiency}} + \underbrace{\frac{\int_{\theta}^{\overline{\theta}} [\omega(s) - \alpha] dF(s)}{\alpha f(\theta)}}_{\text{redistributive motive}}$$

Call $J(\theta) - \theta$ the **distortion term**. Its sign depends on $\int_{\theta}^{\overline{\theta}} \omega(s) - \alpha \, dF(s)$.

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$$\max_{\substack{U^{\mathrm{LF}}(\underline{\theta}) \le \underline{U} \le \underline{\theta} v(q(\underline{\theta})), \\ q \text{ non-decreasing}}} \left[\mathbf{E}[\omega] - \alpha\right] \underline{U} + \alpha \int_{\underline{\theta}}^{\overline{\theta}} \underbrace{\left[J(\theta) v(q(\theta)) - c q(\theta)\right]}_{\text{surplus of virtual type}} \, \mathrm{d}F(\theta),$$

subject to (IR) and (TU).

#4. By envelope theorem, (TU) and (IR) for $\underline{\theta}$ implies (IR) for all θ.

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subject to

with topping up:

$$q(\theta) \geq q^{LF}(\theta) \qquad \forall\, \theta \in [\underline{\theta}, \overline{\theta}], \tag{FOSD}$$

without topping up:

$$\underline{U} + \int_{\underline{\theta}}^{\theta} v(q(s))\mathrm{d}s \geq U^{LF}(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} v(q^{LF}(s))\mathrm{d}s, \qquad \forall\, \theta \in [\underline{\theta}, \overline{\theta}], \tag{SOSD}$$

When To Subsidize

(And When Not To)

When to Subsidize: Scope

Theorem 1. The planner can improve over laissez-faire if and only if the corresponding condition holds:

$$\text{No Topping Up} \;:\; \exists\, \hat{\theta} \in \Theta \;\text{ such that }\; \omega(\hat{\theta}) > \alpha,$$

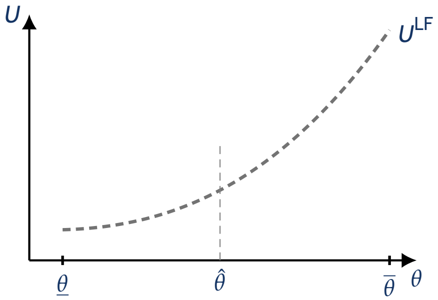
$$\text{Topping Up} \;:\; \exists\, \hat{\theta} \in \Theta \;\text{ such that }\; \mathbf{E}_{\theta}[\omega(\theta) \mid \theta \geq \hat{\theta}] > \alpha.$$

Remark. The scope for subsidies with **topping up** is larger with **no topping up**.

When to Subsidize: Proof Pictures

No Topping Up

$U(\theta) \geq U^{\text{LF}}(\theta)$



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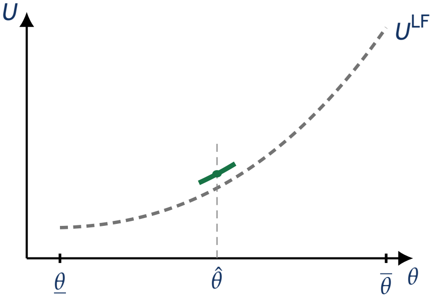

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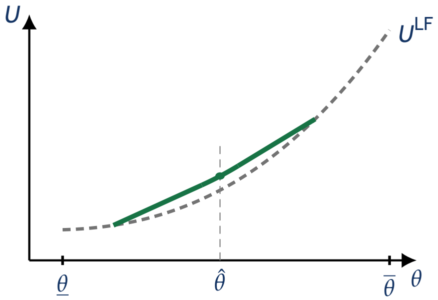

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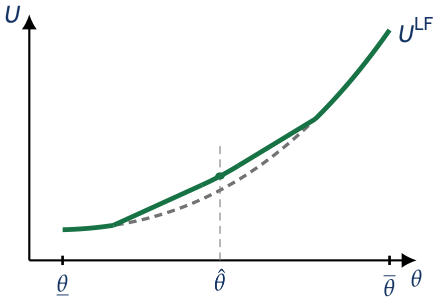

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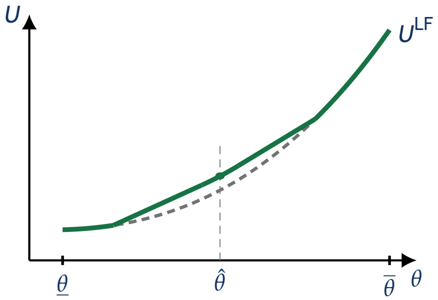


$\exists \hat{\theta} : \ \omega(\hat{\theta}) > \alpha.$

Topping Up

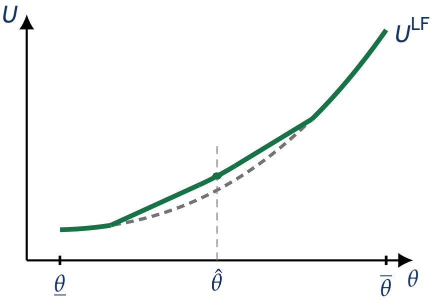
$q(\theta) \geq q^{\text{LF}}(\theta)$

$\Longrightarrow \quad U(\theta) - U^{\text{LF}}(\theta)$ is increasing

When to Subsidize: Proof Pictures

No Topping Up

$U(\theta) \geq U^{LF}(\theta)$



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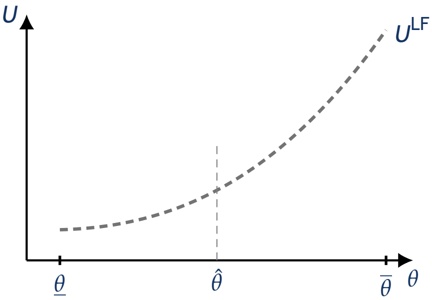

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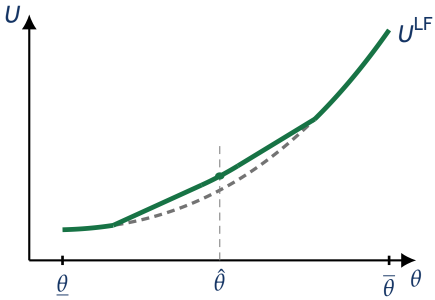

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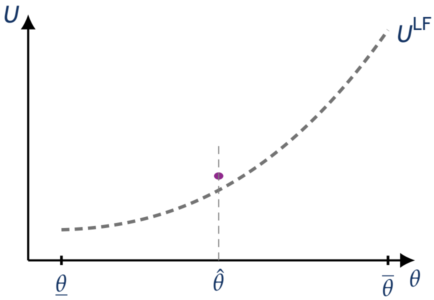

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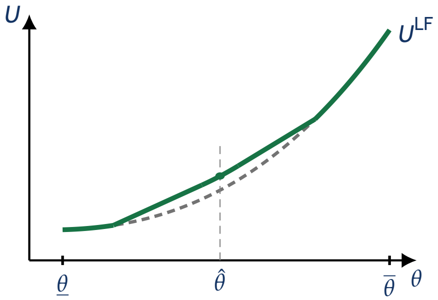

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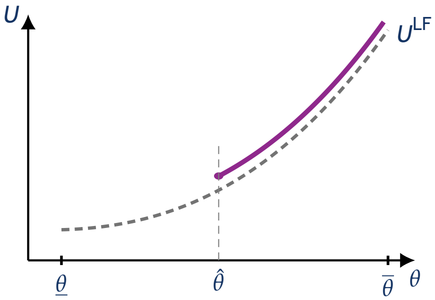

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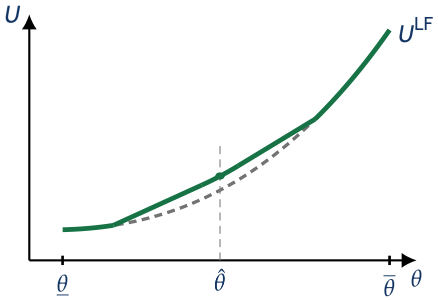

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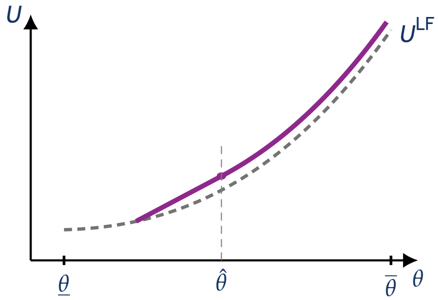

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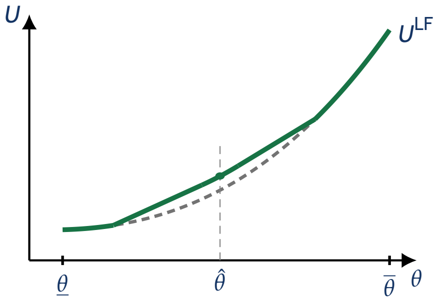

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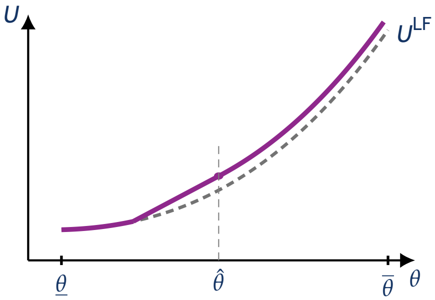


$\exists \hat{\theta} : \mathbf{E}[\omega(\theta) \mid \theta \geq \hat{\theta}] > \alpha.$

When To Subsidize: Economic Implications

Two baseline cases:

"Negative Correlation": $\omega(\theta)$ is decreasing in θ.

- high-demand consumers tend to have lower need for redistribution.
- *e.g.*, food, education, and, if $\omega \propto 1/\text{Income}$, normal goods.

⇝ Wider scope for subsidies without topping up ($\omega(\underline{\theta}) > \alpha$) than with topping up ($\mathbf{E}[\omega] > \alpha$).

"Positive Correlation": $\omega(\theta)$ is increasing in θ.

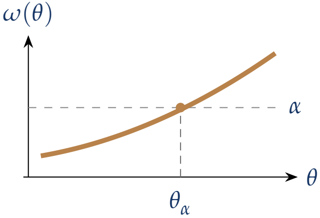
- high-demand consumers tend to have higher need for redistribution.
- *e.g.*, staple foods, public transportation, and, if $\omega \propto 1/\text{Income}$, inferior goods.

⇝ Scope for subsidies is the same with and without topping up ($\omega(\overline{\theta}) > \alpha$).

How To Subsidize

Optimal Design with Positive Correlation

Suppose $\omega(\theta)$ is increasing in θ (e.g., public transport, staple goods).



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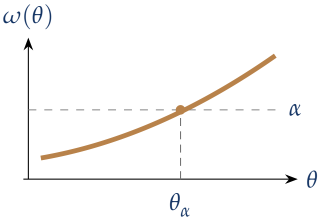

$$J(\theta) = \theta + \frac{\int_\theta^{\overline{\theta}} [\omega(s) - \alpha] \, dF(s)}{\alpha f(\theta)}.$$

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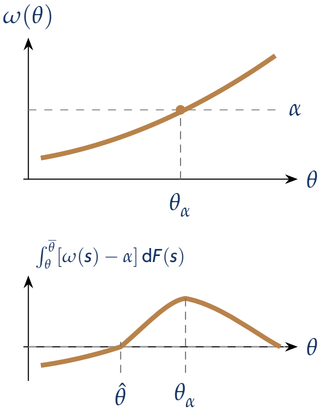

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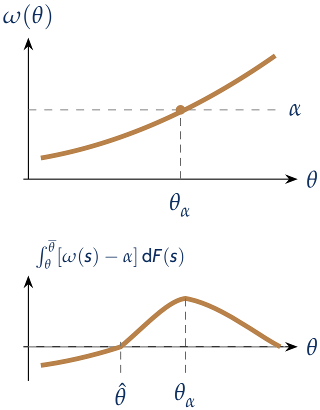

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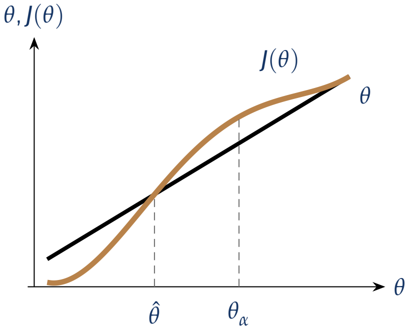


Optimal Design with Positive Correlation

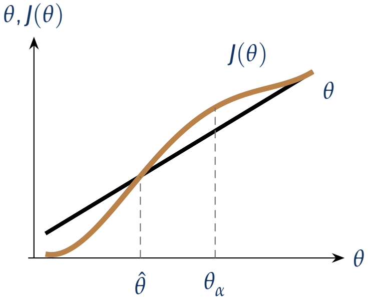
$$\max_{\substack{U^{\text{LF}}(\underline{\theta}) \le \underline{U} \le \underline{\theta} v(q(\underline{\theta})), \\ q \text{ non-decreasing}}} \left[\mathbf{E}[\omega] - \alpha\right] \underline{U} + \alpha \int_{\underline{\theta}}^{\overline{\theta}} \left[J(\theta) v(q(\theta)) - cq(\theta)\right] \mathrm{d}F(\theta) \quad \text{s.t.} \quad \begin{array}{l} U(\theta) \ge U^{\text{LF}}(\theta) \\ q(\theta) \ge q^{\text{LF}}(\theta) \end{array}$$

With positive correlation, the optimal subsidy design **with topping up** and **with no topping up** coincide.

If $\mathbf{E}[\omega] < \alpha$, then

$$q^*(\theta) = \begin{cases} q^{\text{LF}}(\theta) & \text{for } \theta \le \hat{\theta}, \\ D(c, \bar{J}(\theta)) & \text{for } \theta \ge \hat{\theta}, \end{cases}$$

where $\mathbf{E}[\omega(\theta) \mid \theta \ge \hat{\theta}] = \alpha$.



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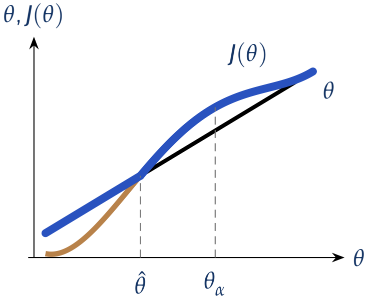

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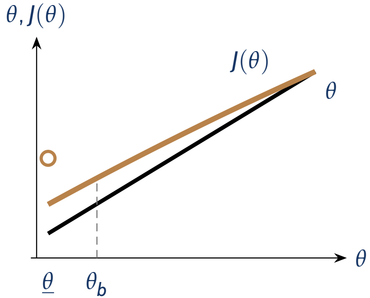

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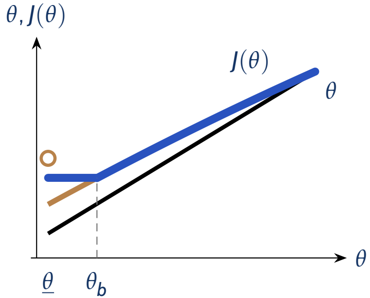

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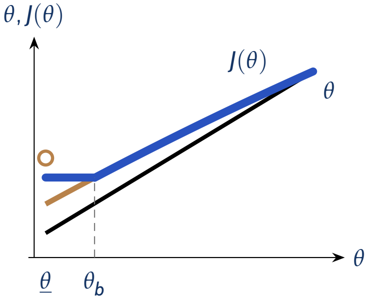

Intuition: Planner's objective decreasing in divergence from J.

(Proof Idea)

Positive Correlation

Suffices to consider **NTU** problem. Let $\lambda(\theta)$ be Lagrange multiplier on constraint $U(\theta) \geq U^{\mathsf{LF}}(\theta)$.

$$\mathcal{L}(q, \overline{U}, \lambda) = ... + \int_{\underline{\theta}}^{\overline{\theta}} \lambda(\theta)[U(\theta) - U^{\mathsf{LF}}(\theta)] \, \mathrm{d}F(\theta)$$

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Complementary slackness $\rightsquigarrow \Lambda(\theta)$ is positive and decreasing in θ, constant where IR is non-binding, and

$$q(\theta) = D\left(J(\theta) + \frac{\Lambda(\theta)}{f(\theta)}, c\right).$$

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Guess and check strong convex duality for Lagrange multipliers $\Lambda(\theta) = 0$ for $\theta > \hat{\theta}$ and $\Lambda(\theta) = -\alpha \int_{\theta}^{\overline{\theta}} [\omega(s) - \alpha] \, \mathrm{d}F(s)$ for $\theta \leq \hat{\theta}$ (positive and decreasing), equivalent to $q = q^*$. $\square$

Economic Implications

With **positive correlation** between ω and θ:

1. The social planner derives **no benefit** from restricting topping up in the private market.

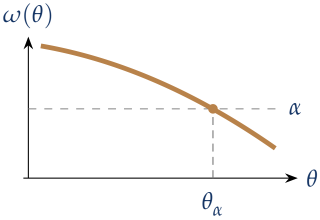
2. Optimal subsidies are **self-targeting**, with benefits flowing only to consumers with the highest need.

3. Social planner **prefers subsidies** to lump-sum cash transfers.

Roughly, positive correlation is the only case in which **topping up** and **no topping up** problems coincide.

Optimal Subsidy Design with Negative Correlation

Suppose $\omega(\theta)$ is decreasing in θ (e.g., normal goods with income-sensitive demand, private transport).



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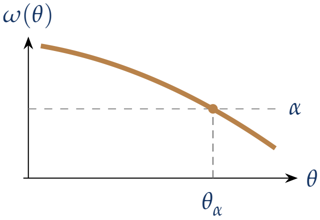

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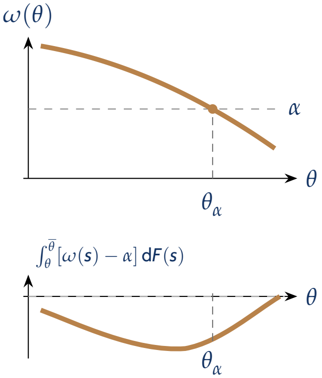

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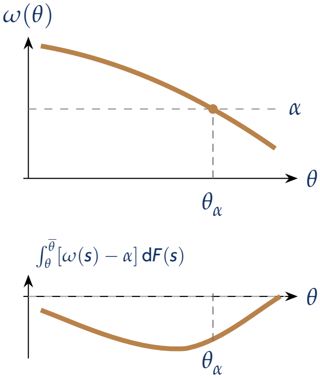

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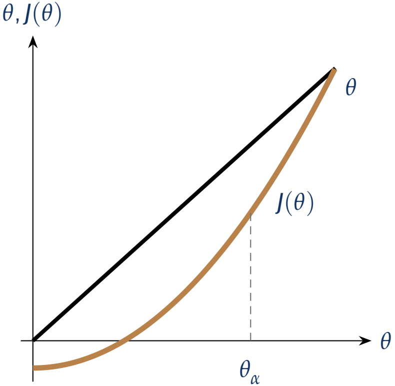
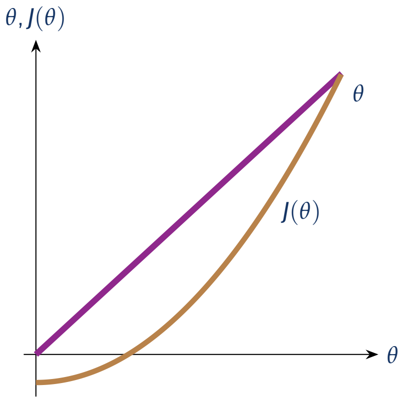


Optimal Subsidy Design with Negative Correlation

Suppose there is negative correlation with $\omega(\underline{\theta}) > \alpha$ and $\omega(\bar{\theta}) < \alpha$. Then the optimal subsidy design differs with **topping up** and with **no topping up**.



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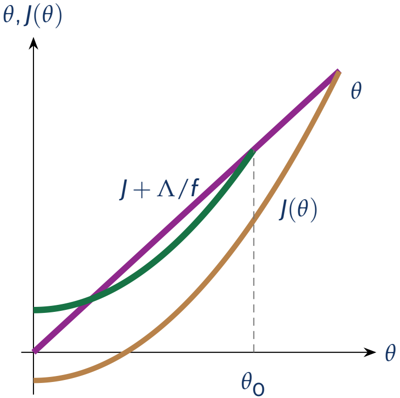


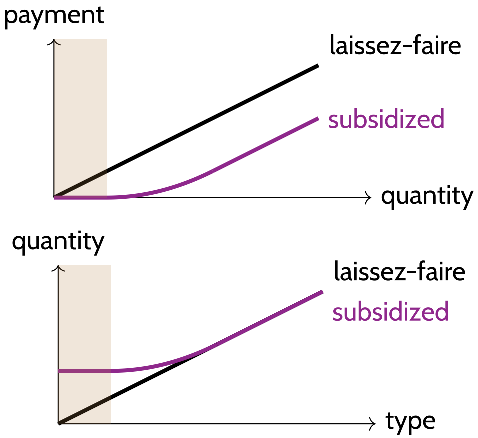
No Topping Up:

- Recall $q^*(\theta) = D(c, \overline{J + \frac{\Lambda}{f}}(\theta))$, where Λ is cumulative Lagrange multiplier on (IR) constraint (nonnegative and decreasing).
- We first show (IR) constraint binds for types $\theta \geq \theta_0 > \hat{\theta}$.
- Implies $\Lambda(\theta) = \mathbf{E}[\alpha - \omega | \theta' \geq \theta]$ for $\theta \geq \theta_0$.
- Then $\Lambda(\theta) = \mathbf{E}[\alpha - \omega | \theta \geq \theta_0]$ is constant for $\theta \leq \theta_0$.
- Find θ_0 such that $U(\theta_0) = U^{LF}(\theta_0)$.

Optimal Subsidy Design

Negative Correlation

With topping up ($\mathbf{E}[\omega] > \alpha$):



Free allocation with partial subsidies up to a cap
(cf. food stamps)

Optimal Subsidy Design

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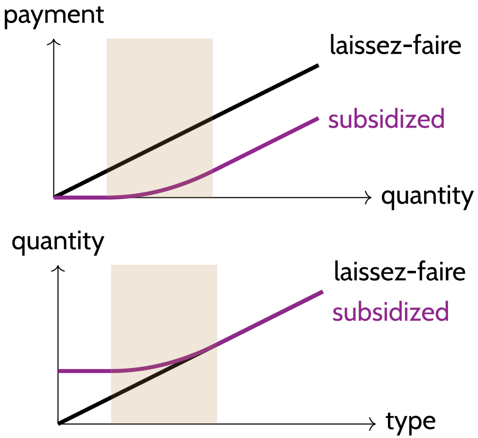

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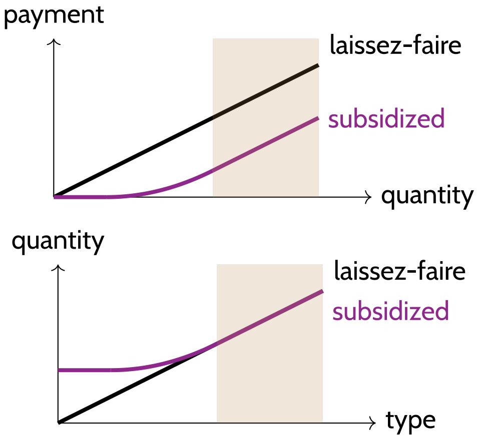

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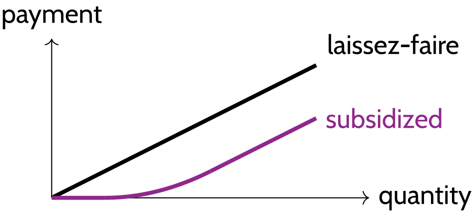

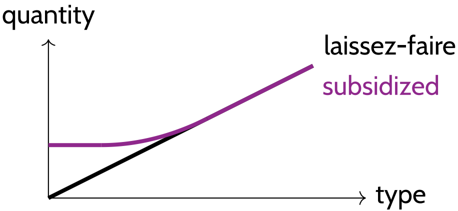

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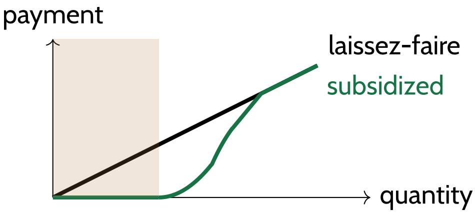

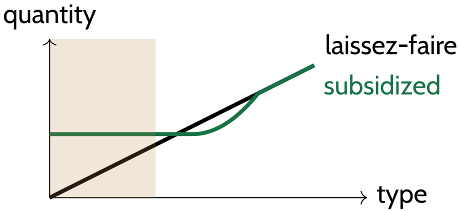

Free allocation and subsidies, intermediate consumption distorted down (cf. public housing)

Optimal Subsidy Design

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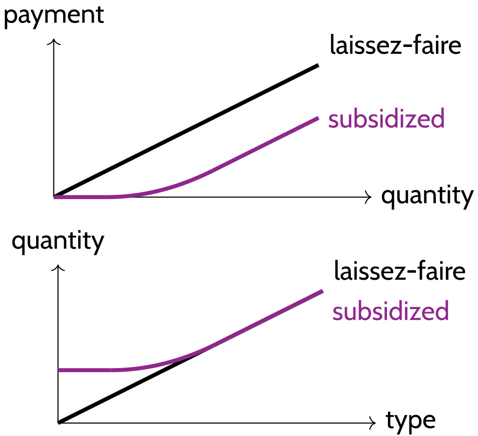

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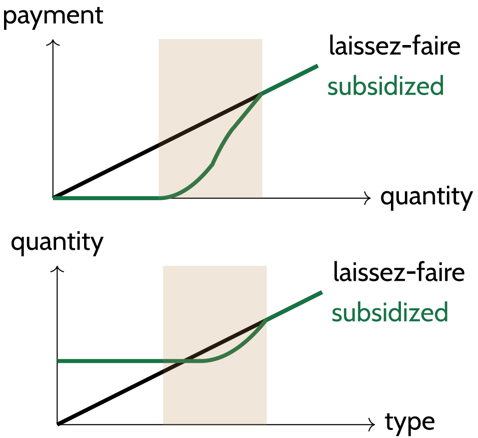

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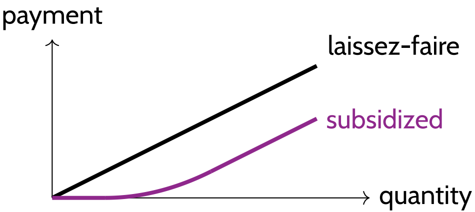

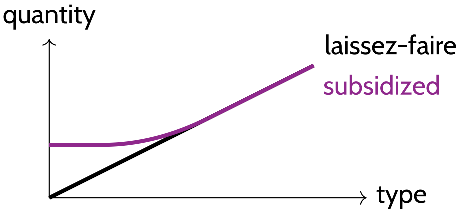

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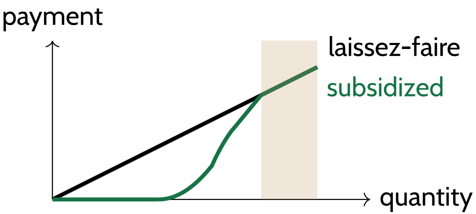

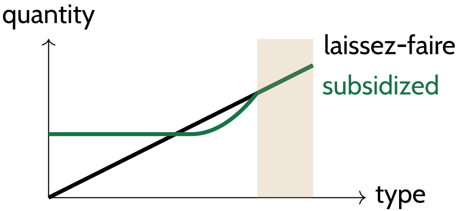

Free allocation and subsidies, intermediate consumption distorted down (cf. public housing)

Economic Implications

With topping up and **negative correlation** between ω and θ:

1. Lump-sum cash transfers are always **more progressive** than subsidies.

2. The optimal subsidy progam is **never linear**, with higher marginal subsidies for low levels of consumption.

  # 2a. Optimal subsidies are “all or none”: active subsidy programs should always incorporate a **free allocation** (“public option”).

  # 2b. If *any* consumer has $\omega < \alpha$, optimal (marginal) subsidies are **capped** in quantity.

Economic Implications

Without topping up and with **negative correlation** between ω and θ:

\# 1. Subsidies are preferred to lump-sum cash transfers, and can be targeted to consumers with high ω.

\# 2. The optimal subsidy program is **never linear**, with higher marginal subsidies for low consumption levels.

   a. The optimal subsidy can involve a **public option** (always if $\mathbf{E}[\omega] \geq \alpha$ and sometimes if $\mathbf{E}[\omega] \leq \alpha$).

   b. If $\mathbf{E}[\omega] \leq \alpha$, high θ (low ω) consumers consume **only** in the private market.

   c. Allocations are always distorted **downwards** for high θ consumers in the subsidy program.

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c. Allocations are always distorted **downwards** for high θ consumers in the subsidy program.

For a fixed α, compared to the optimal subsidy program **with topping up**:

- The set of subsidized consumers is larger.
- Low θ consumers receive a (weakly) larger subsidy, and high θ consumers a (weakly) smaller subsidy.

Discussion

Contrast With "Full" Mechanism Design (No Private Market Constraint)

#1. When should we redistribute in kind?

- **Full design:** always, because we can tax quality consumption of rich to subsidize poor.
- **With topping up:** whenever $\mathbf{E}[\omega(\theta)|\theta \geq \hat{\theta}] > \alpha$ for some $\hat{\theta}$.
- **No topping up:** whenever $\max \omega > \alpha$.

⤳ Participation constraints reduce scope for redistribution, particularly if consumers can top up.

#2. When should we use a free public option?

- **Full design** / **Topping Up:** when $\mathbf{E}[\omega] > \alpha$.
- **No topping up:** when $\mathbf{E}[\omega(\theta)] \geq \alpha$ and sometimes when $\mathbf{E}[\omega] \leq \alpha$.

⤳ Restricting private market access can increase scope for non-market allocations.

Differences In Practice

When? With topping up, scope of intervention larger with positive correlation ($\max \omega > \alpha$) than negative correlation ($\mathbf{E}[\omega] > \alpha$).

In practice, many government programs focused on goods consumed disproportionately by needy (e.g., *Tamween* bread, Indonesian fuel subsidies, dental subsidies) .

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In practice, many government programs focused on goods consumed disproportionately by needy (e.g., *Tamween* bread, Indonesian fuel subsidies, dental subsidies) .

How? Significant differences in marginal subsidy schedules observed in practice:

Larger subsidies for low q

- Food stamps (SNAP)
- Womens, Infants & Children (WIC) Program
- Housing Choice (Section 8) Vouchers
- Lifeline (Telecomm. Assistance) Program
- Public Housing Programs (no topping up)

Larger subsidies for high q

- Public transit fare capping
- Pharmaceutical subsidy programs
- Government-subsidized childcare places.

program details inferior programs

Conclusion

Concluding Remarks

Takeaways for Subsidy Policy:

- Linear subsidies are **never** optimal.
- When and how to subsidize depends on **correlation** between demand and whether topping up is possible/may be restricted:
  - With negative correlation (many goods), the social planner benefits from restricting top-up: e.g., public housing vs. rental assistance. Otherwise, why not lump-sum cash transfers? (“tortilla subsidy” vs. Progresa).
  - Goods with positive correlation are ideal candidates for subsidies (e.g., public transport), but these should have floors for optimal targeting.

Technical Contribution:

- We show how to solve mechanism design problems with FOSD and SOSD constraints caused by type-dependent outside options.
- Similar mechanism design problems arise in other contexts, e.g., subsidy design with other objectives (externalities, paternalism); exclusive contracting (topping up = non-exclusive contracting, no topping up = exclusive contracting.).

Fin

Appendix

Assumption: No Lump-Sum Cash Transfers

Note: This constraint only binds if $\mathbf{E}_\theta[\omega(\theta)] > \alpha$.

Possible reasons:

- **Institutional**: subsidies designed by government agency without tax/transfer powers.
- **Political**: Liscow and Pershing (2022) find U.S. voters prefer in-kind redistribution to cash transfers.
- **Household Economics**: Currie (1994) finds in-kind redistribution has stronger benefits for children than cash transfer programs.
- **Pedagogical**: to contrast when the assumption is binding ($\rightsquigarrow$ cash transfers preferred to subsidies) versus non-binding (*vice versa*).
- **Model**: without NLS constraint, the social planner would want to make unbounded cash transfers when $\mathbf{E}[\omega] > \alpha$.

return

Topping Up $\Longleftarrow$ Lower-Bound (1/2)

Suppose $q(\theta) \geq q^{\text{LF}}(\theta)$. We want to show total subsidies $S(z)$ is increasing in z.

1. $t(\underline{\theta}) \leq cq(\underline{\theta})$ by (IR):

$$t(\underline{\theta}) \leq \underline{\theta}v(q(\underline{\theta})) - \underline{\theta}v(q^{\text{LF}}(\underline{\theta})) + cq^{\text{LF}}(\underline{\theta}),$$

and $\underline{\theta}v(q^{\text{LF}}(\underline{\theta})) - cq^{\text{LF}}(\underline{\theta}) \geq \underline{\theta}v(q(\underline{\theta})) - cq(\underline{\theta})$ by definition of q^{LF}, so $t(\underline{\theta}) \leq cq(\underline{\theta})$.

Topping Up $\Longleftarrow$ Lower-Bound (2/2)

2. The *marginal* price of any units purchased is no greater than c by (IC):

$$t(\theta') - t(\theta) = \left[\theta' v(q(\theta')) - U(\underline{\theta}) - \int_{\underline{\theta}}^{\theta'} v(q(s))\, ds\right] - \left[\theta v(q(\theta)) - U(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} v(q(s))\, ds\right]$$

$$= \theta' v(q(\theta')) - \theta v(q(\theta)) - \int_{\theta}^{\theta'} v(q(s))\, ds$$

$$= \int_{\theta}^{\theta'} s v'(q(s))\, dq(s).$$

But if $q(\theta) \geq q^{\mathrm{LF}}(\theta)$, then concavity of v implies $v'(q(\theta)) \leq v'(q^{\mathrm{LF}}(\theta)) = c/\theta$, so $t(\theta') - t(\theta) \leq c[q(\theta') - q(\theta)]$.

▸ return

Characterizing the Optimal Subsidy With Topping Up

Theorem. The optimal allocation rule is unique, continuous and satisfies

$$q^*(\theta) = \begin{cases} D\left(c, \overline{J|_{[\underline{\theta},\theta_\alpha]}}(\theta)\right) & \text{for } \theta \leq \theta_\alpha \\ q^{\text{LF}}(\theta) & \text{for } \theta \geq \theta_\alpha, \end{cases}$$

where θ_α is defined by

$$\theta_\alpha = \inf\left\{\theta \in \Theta : \overline{J|_{[\underline{\theta},\theta]}}(\theta) \leq \theta\right\}.$$

Intuition: there exists a type $\theta_\alpha \in \Theta$ (possibly $\underline{\theta}$ or $\overline{\theta}$) such that

$$q^*(\theta) > q^{\text{LF}}(\theta) \text{ for all } \theta < \theta_\alpha, \text{ and}$$
$$q^*(\theta) = q^{\text{LF}}(\theta) \text{ for all } \theta \geq \theta_\alpha.$$

▸ Ironing

Solving for the Optimal Mechanism

return to summary

$$\max_{q} \alpha \int_{\underline{\theta}}^{\overline{\theta}} \left[J(\theta)v(q(\theta)) - cq(\theta)\right] \, \mathrm{d}F(\theta),$$

s.t. q nondecreasing and $q(\theta) \geq q^{\mathrm{LF}}(\theta)$.

Solving for the Optimal Mechanism

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$$\max_{q} \alpha \int_{\underline{\theta}}^{\overline{\theta}} [J(\theta)v(q(\theta)) - cq(\theta)] \, \mathrm{d}F(\theta),$$

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Guess 1: Pointwise maximizer

$$q(\theta) = (v')^{-1}\left(\frac{c}{J(\theta)}\right) = D(c, J(\theta)).$$

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q nondecreasing $\iff J(\theta)$ nondecreasing.
$q \geq q^{\mathrm{LF}} \iff D(c, J(\theta)) \geq D(c, \theta) \iff J(\theta) \geq \theta$.

Solving for the Optimal Mechanism

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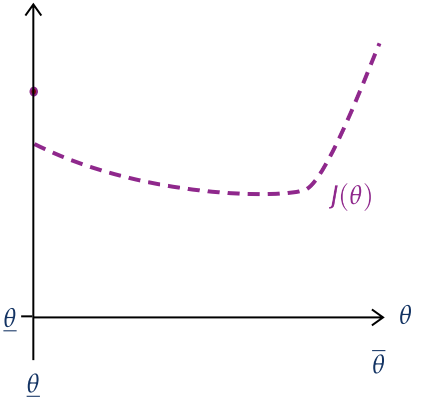

$J(\theta)$ may be non-monotone.

Solving for the Optimal Mechanism

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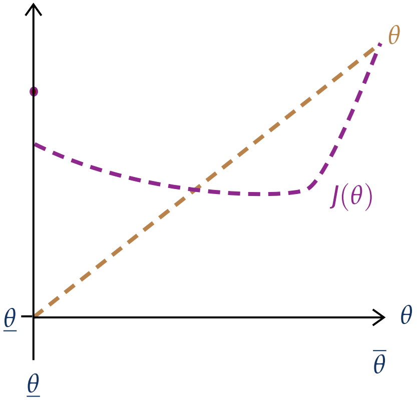

$J(\theta)$ may be smaller than θ.

Solving for the Optimal Mechanism

return to summary

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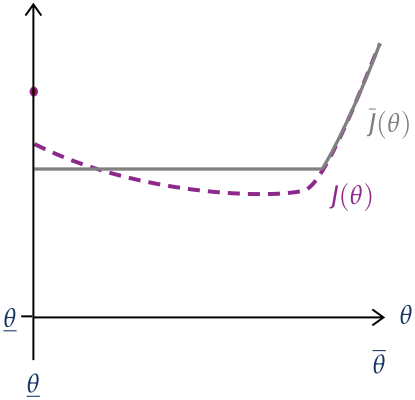
s.t. q nondecreasing and $q(\theta) \geq q^{\text{LF}}(\theta)$.

Guess 2: Relaxing the (LB) constraint

Toikka (2011); Akbarpour, Dworczak, Kominers (2021)

$$\rightsquigarrow q(\theta) = (v')^{-1}\left(\frac{c}{\bar{J}(\theta)}\right) = D(c, \bar{J}(\theta)),$$

where $\bar{J}$ is ironing of J, pooling types in any non-monotonic interval of J at its F-weighted average.



Ironing deals with non-monotonicity.

Ironing

Solving for the Optimal Mechanism

return to summary

$$\max_q \alpha \int_{\underline{\theta}}^{\overline{\theta}} \left[J(\theta) v(q(\theta)) - cq(\theta) \right] \, \mathrm{d}F(\theta),$$

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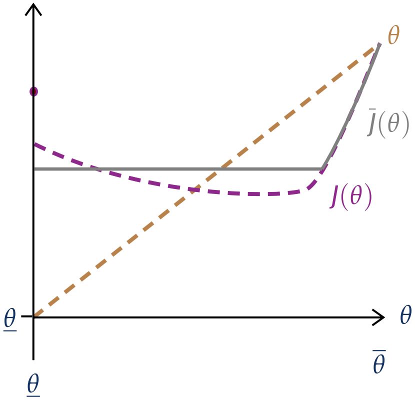

But not lower-bound constraint $\rightsquigarrow$ interaction.

Ironing

Solving for the Optimal Mechanism

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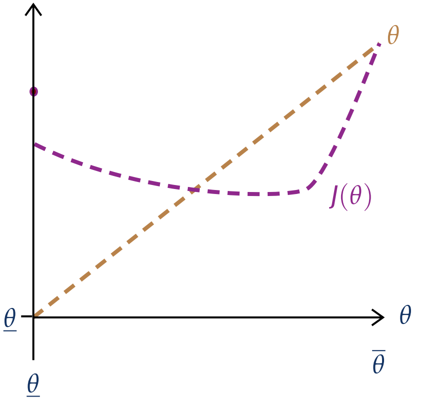
s.t. q nondecreasing and $q(\theta) \geq q^{\mathrm{LF}}(\theta)$.

Guess 3: Our approach

Suppose solution is of the form

$$q(\theta) = D(c, H(\theta)).$$

Feasibility requires H to be nondecreasing and satisfy $H(\theta) \geq \theta$.



Need to identify nondecreasing $H \geq \theta$.

Ironing

Characterizing the Optimal Subsidy Allocation

Theorem. The optimal allocation rule is unique, continuous and satisfies

$$q^*(\theta) = D(c, H(\theta)),$$

where the subsidy type $H(\theta)$ is defined by

$$H(\theta) := \begin{cases} \overline{J|_{[\underline{\theta},\theta_\alpha]}}(\theta) & \text{for } \theta \leq \theta_\alpha \\ \theta & \text{for } \theta \geq \theta_\alpha, \end{cases}$$

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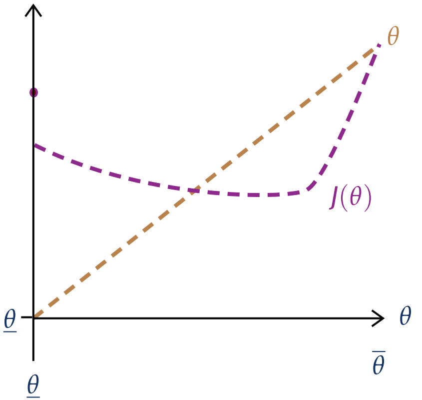

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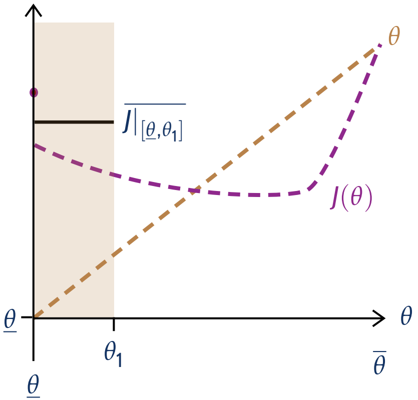

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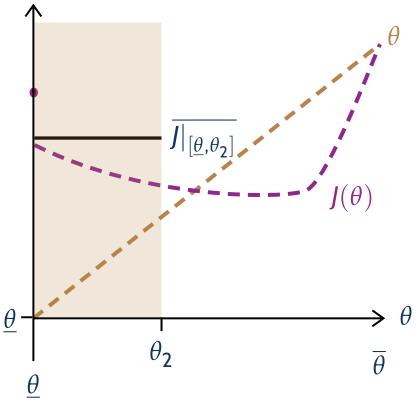

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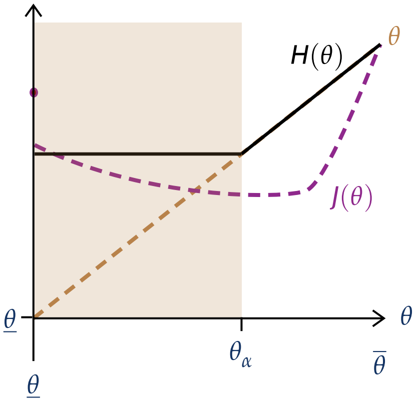

construction $\rightsquigarrow$ pooling condition and continuity

▸ Ironing

Verifying H from Theorem 2

Because $q^*(\theta) = D(c, H(\theta))$, for any feasible q

$$\int_\Theta \underbrace{\left[H(\theta)v(q^*(\theta)) - cq^*(\theta)\right]}_{\text{surplus of type } H(\theta) \text{ at } D(c,H(\theta))} \mathrm{d}F(\theta) \geq \int_\Theta \underbrace{\left[H(\theta)v(q(\theta)) - cq(\theta)\right]}_{\text{surplus of type } H(\theta) \text{ at } q(\theta)} \mathrm{d}F(\theta).$$

skip

Verifying H from Theorem 2

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$$\int_\Theta \underbrace{[H(\theta)v(q^*(\theta)) - cq^*(\theta)]}_{\text{surplus of type } H(\theta) \text{ at } D(c,H(\theta))} \,\mathrm{d}F(\theta) \geq \int_\Theta \underbrace{[H(\theta)v(q(\theta)) - cq(\theta)]}_{\text{surplus of type } H(\theta) \text{ at } q(\theta)} \,\mathrm{d}F(\theta).$$

We want to show, for any feasible q

$$\underbrace{\int_\Theta [J(\theta)v(q^*(\theta)) - cq^*(\theta)] \,\mathrm{d}F(\theta)}_{\text{objective at } q^*} \geq \underbrace{\int_\Theta [J(\theta)v(q(\theta)) - cq(\theta)] \,\mathrm{d}F(\theta)}_{\text{objective at feasible } q}.$$

skip

Verifying H from Theorem 2

Because $q^*(\theta) = D(c, H(\theta))$, for any feasible q

$$\int_\Theta \underbrace{[H(\theta)v(q^*(\theta)) - cq^*(\theta)]}_{\text{surplus of type } H(\theta) \text{ at } D(c,H(\theta))} \, dF(\theta) \geq \int_\Theta \underbrace{[H(\theta)v(q(\theta)) - cq(\theta)]}_{\text{surplus of type } H(\theta) \text{ at } q(\theta)} \, dF(\theta).$$

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Subtracting, it suffices to show, for any feasible q

$$\int_\Theta [J(\theta) - H(\theta)][v(q^*(\theta)) - v(q(\theta))] \, dF(\theta) \geq 0.$$

skip

Verifying the Variational Inequality

To show $\int_\Theta [J(\theta) - H(\theta)][v(q^*(\theta)) - v(q(\theta))]\, dF(\theta) \geq 0$.

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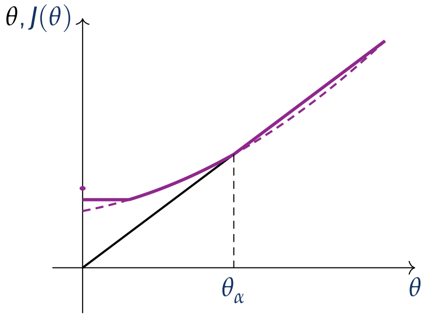
There are three possibilities for H, partitioning Θ into intervals:



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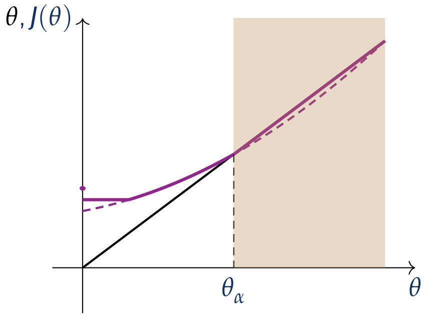
1. $H(\theta) = \theta$: by construction $J(\theta) \leq \theta = H(\theta)$ and $v(q(\theta)) \geq v(q^*(\theta)) \rightsquigarrow$ integrand ≥ 0.



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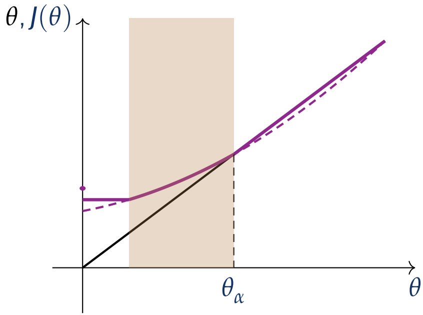

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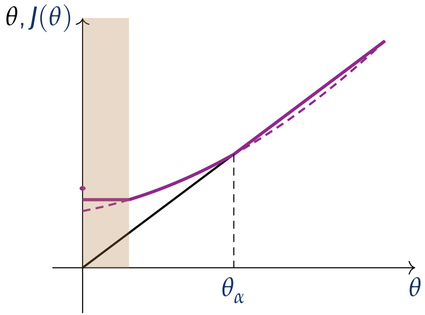
#3. $H(\theta) = \overline{J|_{[\underline{\theta},\theta_\alpha]}}(\theta) \neq J(\theta)$:
technical lemma $\rightsquigarrow$ on any such interval Θ_i, $H = \overline{J|_{\Theta_i}}$
$\rightsquigarrow$ optimality of $D(c, H(\theta))$ in problem on Θ_i *without* (LB)
$\Longrightarrow$ same variational inequality characterizes optimality. $\square$



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Summing Up

Proof approach:

- Guess form of solution $q^*(\theta) = D(c, H(\theta))$.
- Identify $H(\theta)$ which is continuous, $\geq \theta$, and satisfies the **pooling condition**.
- Verify optimality using **variational inequalities**.

Same method of solution works for general $\omega \rightsquigarrow$ see paper.

▸ Generalization

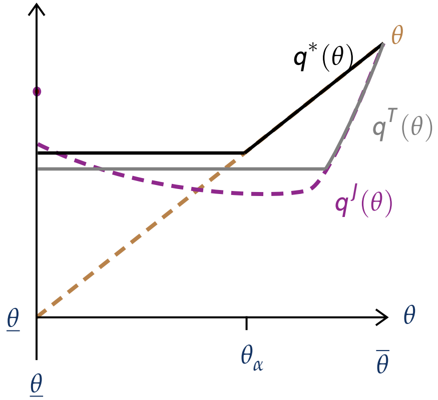
Role of The Private Market

Comparing optimum with and without (LB) constraint, $q^*(\theta)$ can exceed $q^T(\theta)$ for *all* types.

$\rightsquigarrow$ Inability to tax can cause upward distortion, even for consumers who would be subsidized in the absence of the (LB) constraint.

It is not optimal to calculate optimal subsidy/tax and set taxes to zero.

Highlights distinction from Mirrleesian marginal approach (FOC $\not\rightsquigarrow$ optimum).



Characterization of Optimal Mechanism

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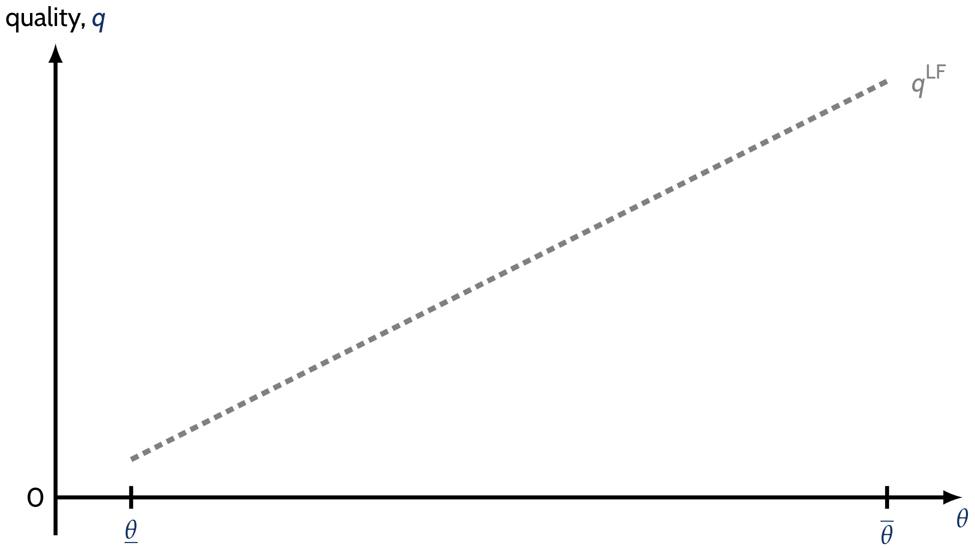

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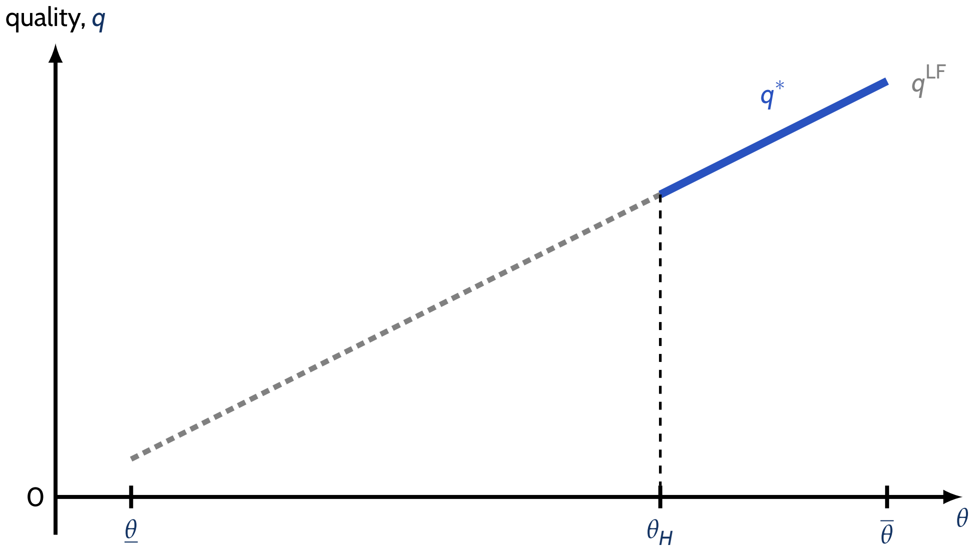

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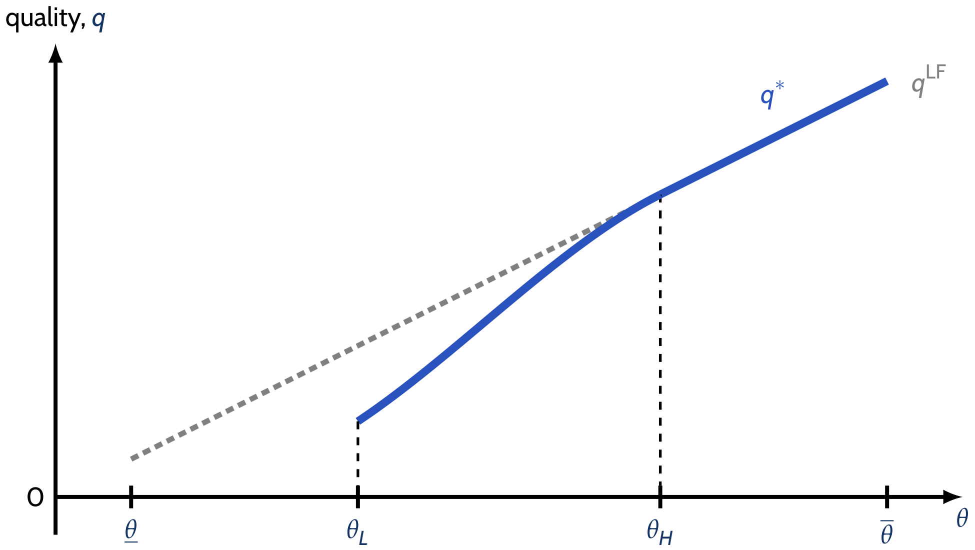

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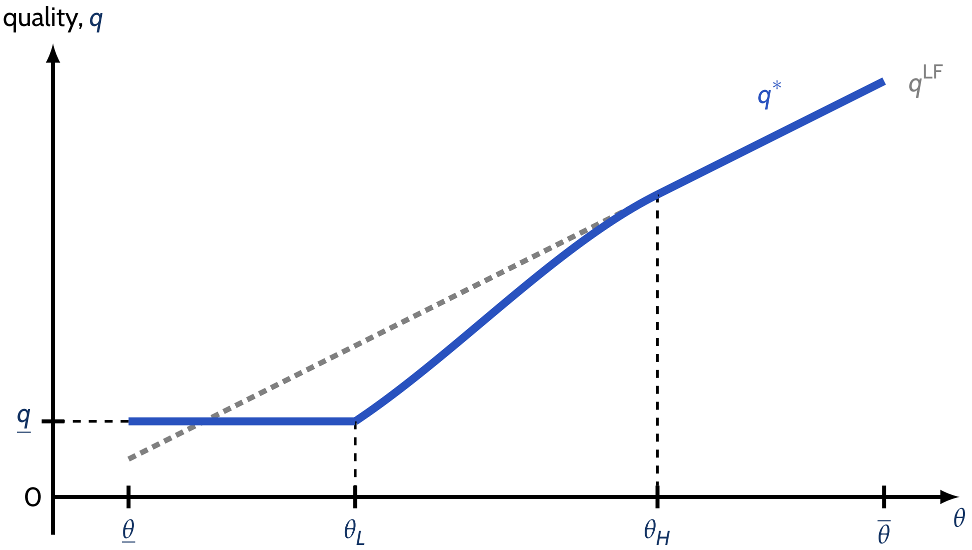

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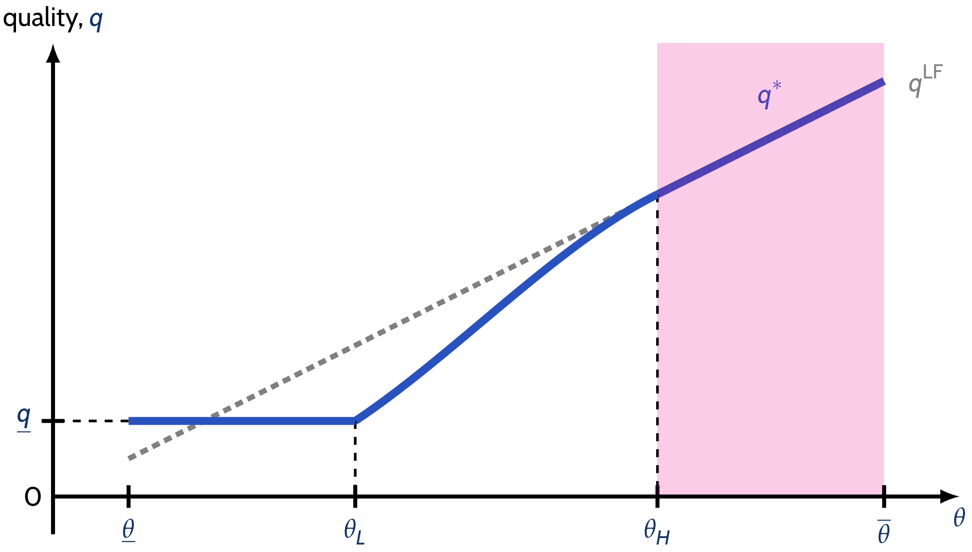

Ⓐ Which consumers go to the private market?

Theorem 2(a). Under the optimal mechanism:

- If $\mathbf{E}[\omega] \leq \alpha$, then there exists $\mu^* \geq 0$ such that the (IR) constraint binds exactly for consumers with types in $[\theta_H, \overline{\theta}]$, where

$$\theta_H := \max \left\{ \theta \in [\underline{\theta}, \overline{\theta}] : \int_{\underline{\theta}}^{\theta} [\alpha - \omega(s)] \, dF(s) + \mu^* \leq 0 \right\}.$$

- If $\mathbf{E}[\omega] > \alpha$, then $\theta_H = \overline{\theta}$.

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$\theta \mapsto \int_{\underline{\theta}}^{\theta} [\alpha - \omega(s)] \; \mathrm{d}F(s)$

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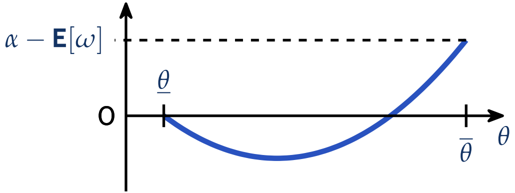

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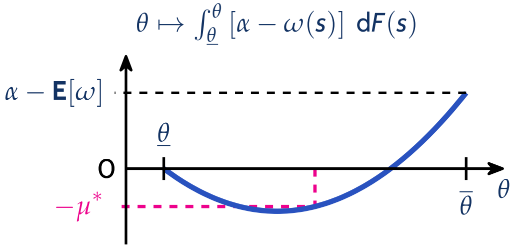

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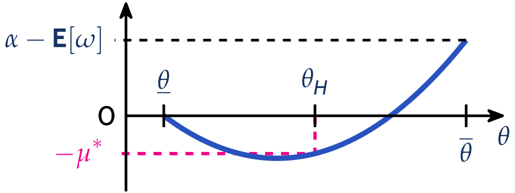

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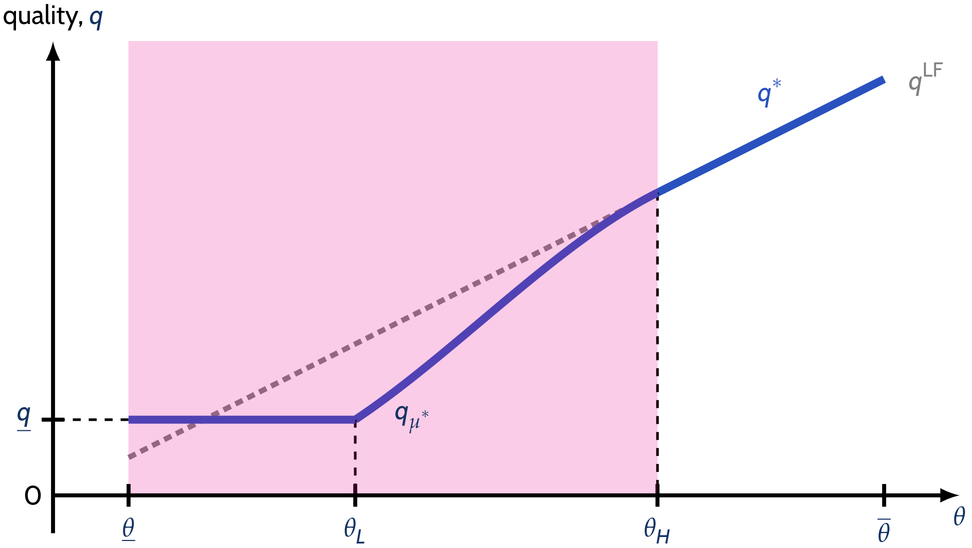
- If $\mathbf{E}[\omega] > \alpha$, then $\theta_H = \overline{\theta}$ (this holds even if $\omega(\overline{\theta}) < \alpha$!).

B Which consumers benefit from in-kind redistribution?



(B) Which consumers benefit from in-kind redistribution?

Theorem 2(b). For any $\mu \geq 0$, define

$$q_\mu(\theta) := D(c, \overline{H_\mu}(\theta)), \quad \text{where } H_\mu(\theta) := \frac{\theta}{c} + \frac{\mu\underline{\theta} \cdot \delta_{\theta=\underline{\theta}} + \mu + \int_{\underline{\theta}}^{\theta} [\alpha - \omega(s)] \, \mathrm{d}F(s)}{\alpha c f(\theta)},$$

$$\theta_H(\mu) := \begin{cases} \max\left\{\theta \in [\underline{\theta}, \overline{\theta}] : \int_{\underline{\theta}}^{\theta} [\alpha - \omega(s)] \, \mathrm{d}F(s) + \mu \leq 0\right\} & \text{if } \mathbf{E}[\omega] \leq \alpha, \\ \overline{\theta} & \text{if } \mathbf{E}[\omega] > \alpha. \end{cases}$$

Under the optimal mechanism, consumers with types in $[\underline{\theta}, \theta_H(\mu^*)]$ consume $q^*(\theta) = q_{\mu^*}(\theta)$, where

$$\mu^* := \min\left\{\mu \in \mathbb{R}_+ : \int_{\underline{\theta}}^{\theta_H(\mu)} v(q_\mu(s)) \, \mathrm{d}s + \underline{\theta} v(q_\mu(\underline{\theta})) - U^{\mathrm{LF}}(\theta_H(\mu)) \geq 0\right\}.$$

Optimal Subsidy Design Without Topping Up

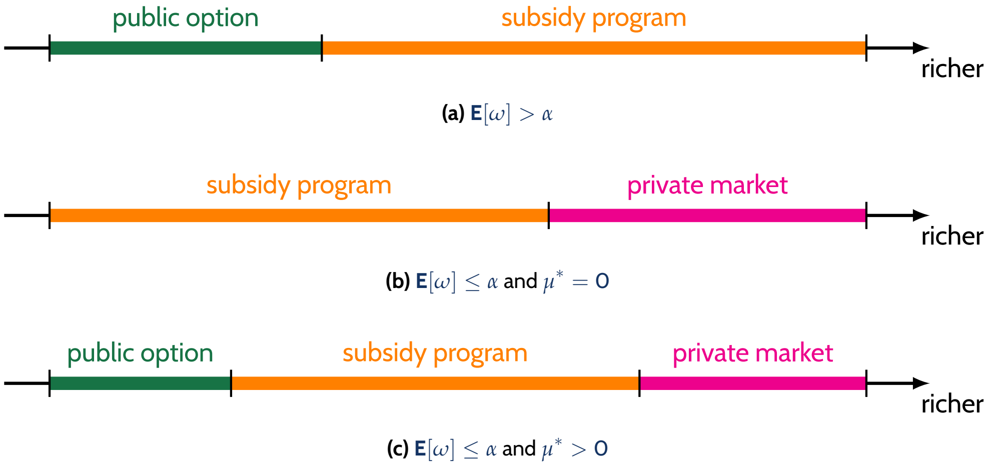


Figure Optimal in-kind redistribution programs under negative correlation.

▸ Proof Sketch

Comparative Statics of Subsidies

Question: How do optimal subsidies change when

(a) the social planner's desire to redistribute to each consumer increases?

(b) the correlation between demand and welfare weight increases?

(c) the marginal cost of production decreases?

▸ Details

Comparative Statics of Subsidies

Question: How do optimal subsidies change when

(a) the social planner's desire to redistribute to each consumer increases?

(b) the correlation between demand and welfare weight increases?

(c) the marginal cost of production decreases?

Details

Short Answer: Each cause the optimal subsidy program to be more generous.

But (a) and (b) cause $J(\theta)$ to increase for each $\theta \rightsquigarrow$ a larger set of consumers subsidized. (c) does not.

Equilibrium Effects

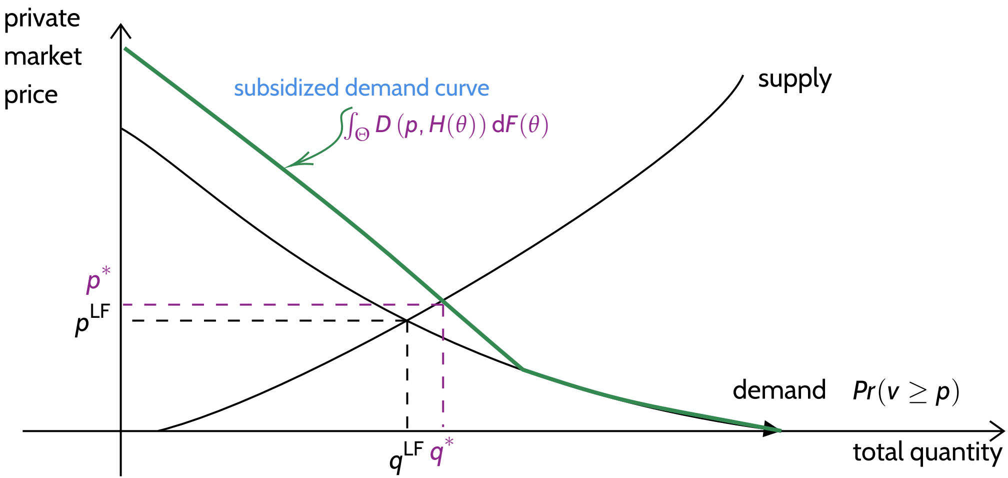
Our baseline model shuts down equilibrium effects of government subsidies on private market prices.

Empirical evidence of price effects from government subsidy programs, *e.g.*:

- public housing (Diamond and McQuade, 2019; Baum-Snow and Marion, 2009)
- pharmaceuticals (Atal et al., 2021)
- public schools (Dinerstein and Smith, 2021)
- school lunches (Handbury and Moshary, 2021)

Equilibrium Effects

Our results extend directly to imperfectly elastic supply curves:



Private Market Taxation

Our baseline model assumes the planner cannot tax the private market.

Taxation of private market reduces consumers' outside option, relaxing the (LB) constraint. If taxation is costly (*e.g.*, because of distortions on ineligible consumers):

Proposition. Suppose the planner faces a convex cost $\Gamma(\tau)$ for taxation of the private market. Then there exists an optimal tax level τ^* and subsidy program for eligible consumers satisfying

$$q^*(\theta) = D(c, H_{\tau^*}(\theta)),$$

where $H_{\tau^*}(\theta) \leq H(\theta)$.

Return to Model

Budget Constraints and Endogenous Welfare Weights

In our baseline model, $\omega(\cdot)$ and α are taken exogenously.

Our model can be extended to allow weights to be endogenous (cf. Pai and Strack, 2024):

- $\alpha \iff$ Lagrange multiplier on the social planner's budget constraint.
- $\omega(\theta) \iff$ the marginal value of money for a consumer with concave preferences

$$\varphi\left(\theta v(q) + I - t\right),$$

and income $I \sim G_\theta$, known but not observed by the social planner, then

$$\omega(\theta) = \mathbf{E}_{I \sim G_\theta}\left[\varphi'\left(\theta v(q(\theta)) + I - t(\theta)\right)\right].$$

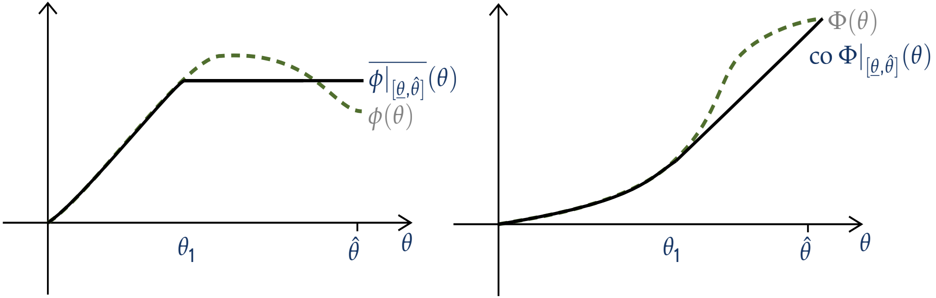
▸ Return to Model

Ironing

Let ϕ be a (generalized) function and $\Phi : \theta \mapsto \int_{\underline{\theta}}^{\theta} \phi(s)\, \mathrm{d}F(s)$. Then $\overline{\phi}$ is the monotone function satisfying

$$\text{for all } \theta \in [\underline{\theta}, \hat{\theta}], \qquad \int_{\underline{\theta}}^{\theta} \overline{\phi}(s)\, \mathrm{d}F(s) = \mathrm{co}\, \Phi(\theta).$$

Intuitively, $\overline{\phi}$ replaces non-monotone intervals of ϕ with F-weighted averages.



▸ Proof ▸ Statement