Strong Monotonicity and Perturbation-Proofness of Exchange Economies

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Abstract

I study the price impact of small perturbations to Walrasian equilibrium in exchange economies, as might arise from agents' misreports, changes in the supply vector, or changes in the set of participants. A sequence of markets is *perturbation-proof* if the price impact of any perturbation is inversely proportional to the number of agents. Perturbation-proofness implies good large-market incentive properties of Walrasian equilibrium and robustness of prices to small misspecifications. Replica economies are perturbation-proof if and only if the base economy's demand correspondence is *strongly monotone*. When buyers' types are drawn identically and independently from a distribution with a strongly monotone expected demand correspondence, the resulting sequence of economies is perturbation-proof with high probability.

Keywords: Approximate incentive-compatibility, General equilibrium, Market design, Perturbation analysis, Prices, Strong convexity, Strong monotonicity

JEL Codes: C610, D400, D440, D470, D500, D510.

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1 Introduction

Consider a nested sequence of quasilinear markets indexed by the number of agents, N. Suppose each market is slightly perturbed by changing an agent's report; adjusting supply; or adding or removing some agents. When does the price effect of any such perturbation diminish rapidly with market size, specifically at a rate inversely proportional to N?

I define such a sequence of markets to be *perturbation-proof.* Perturbation-proofness implies good incentives in Walrasian mechanisms: in a perturbation-proof sequence of markets, the benefit of unilateral misreporting in any Walrasian mechanism disappears at that same fast rate.

Walrasian mechanisms hold interest for two key reasons: they serve as a canonical model of many real markets, and market designers routinely implement mechanisms that choose or approximate Walrasian equilibria. A concern with these mechanisms, known since Hurwicz (1972), is their vulnerability to strategic manipulation by agents with private information. Although price-taking behavior typically emerges in the limit as $N \to \infty$ (Roberts and Postlewaite, 1976), real-world markets are finite and most existing literature on rates of convergence to price-taking apply only in narrow preference domains.[1] This paper provides a general condition on demand, *strong monotonicity*, that implies perturbation-proofness and the rapid convergence of reporting incentives.[2] These results facilitate the assessment of Walrasian mechanisms' performance with respect to strategic incentives in new economic environments.

Strong monotonicity is a condition on the responsiveness of demand to price changes. With a single good, strong monotonicity requires there to be a lower bound on the slope of the demand curve. The general definition is presented in Section 3. Markets in which all buyers have strongly monotone demand (Theorem 1) and replica economies of a strongly monotone base economy (Theorem 2) are perturbation-proof. With one good, the intuition is clear: as the number of agents with strongly monotone demand grows, the market demand curve becomes steeper, causing small movements in the supply curve[3] to lead to progressively smaller movements in the equilibrium price. In replica economies, strong monotonicity is not only sufficient but also *necessary* for perturbation-proofness.

Next, I study an independent private valuations (IPV) model of markets, where buyer types are drawn identically and independently according to a distribution with strongly monotone *expected*

[1] Such as the unit-demand double auction of Satterthwaite and Williams (1989) and the linear-quadratic models surveyed by Rostek and Yoon (2020)

[2] Strong monotonicity and the related notion of strong convexity (discussed in Appendix A) are standard tools in perturbation analysis and algorithmic optimization.

[3] In Proposition 1, I show that all perturbations in the first paragraph may be thought of as changes in supply.

demand. Theorem 3 shows that the realized sequence of markets is perturbation-proof with high probability and in expectation over draws of the market. This implies that the ex post benefit of *any* misreport by a single agent is $O(1/N^{1-\varepsilon})$ for any $\varepsilon > 0$ with probability approaching 1 as N tends to infinity.[4] A corollary is that the interim expected benefit of the optimal misreport is $O(1/N^{1-\varepsilon})$, which is faster than the $O(1/N^{\frac{1}{2}-\varepsilon})$ rate of interim incentives implied by the "strategy-proofness in the large" results of Azevedo and Budish (2019).

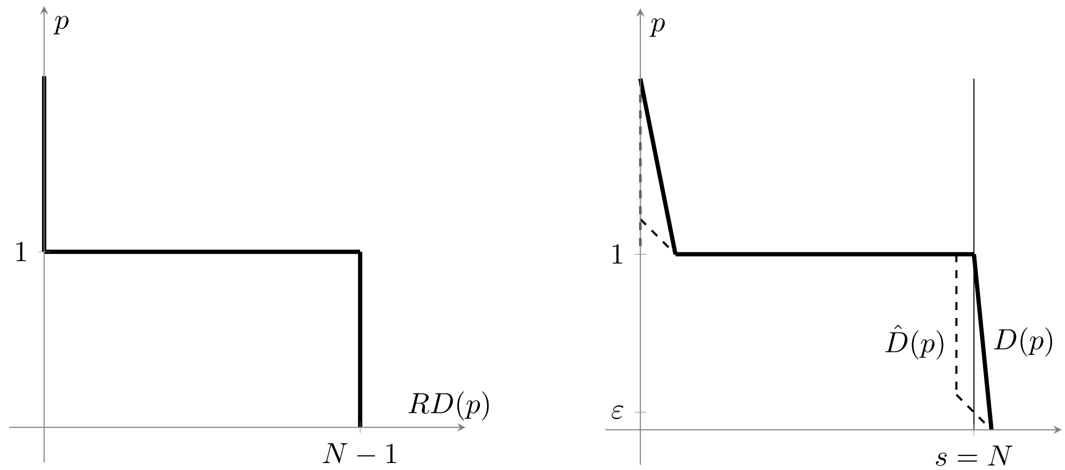
I then apply these results to economic models with indivisibilities, where strong monotonicity of expected demand reduces to a condition on the prices at which demand changes—not the size of these demand changes, which is bounded below by the size of the indivisibility. In this setting, I show that strong monotonicity holds if the probability that an agent's demand changes between any two prices grows at least proportionally to the distance between the two prices. This is interpreted as a condition on *variety* in the possible preferences of buyers and *uncertainty* about the prices associated with demand changes (these notions are formalized below). These results are applied to derive new incentive properties of the Walrasian mechanism in a market with complementarities in buyers' preferences.

Examples This section contrasts two sequences of markets: one where a buyer can significantly influence prices regardless of the market size, and another where each buyer's expected influence is $O(1/N)$. This comparison illustrates how the demand curve's slope affects the price impact of small perturbations.

Example 1.1. Consider an economy with a single consumption good and N buyers. The first $N-1$ buyers have unit demand for the good with value 1, while the N^{th} buyer's demand as a function of price is $D_N(p) = \max\{2-p, 0\}$. The mechanism designer uses a Walrasian mechanism in this market.

Suppose the supply is N and all buyers report their preferences truthfully, so $D(p) := \sum_{n=1}^{N} D_n(p)$. In this scenario, $D(p) = N$ at $p = 1$, so buyer N receives one unit in equilibrium at a price of 1. However, if buyer N misreports and claims to have demand $\hat{D}_N(p) = \max\{2 - p/\varepsilon, 0\}$, the equilibrium condition $\hat{D}(p) := \sum_{n=1}^{N} D_n(p) + \hat{D}_n(p) = N$ is satisfied only at $p = \varepsilon$, so buyer N can drive the price arbitrarily close to zero. That is, the set of Walrasian equilibrium prices attainable to buyer N by some report is $[0, 1]$. Since the buyer receives a single unit of the good under any of these reports, any report that lowers the price benefits the buyer.

[4] Recall Knuth's 1976 big O notation: given $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, $f(x) = O(g(x))$ if $\limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty$.



(a) Residual demand (without buyer N)

(b) Truthful and perturbed demand (dashed) with buyer N

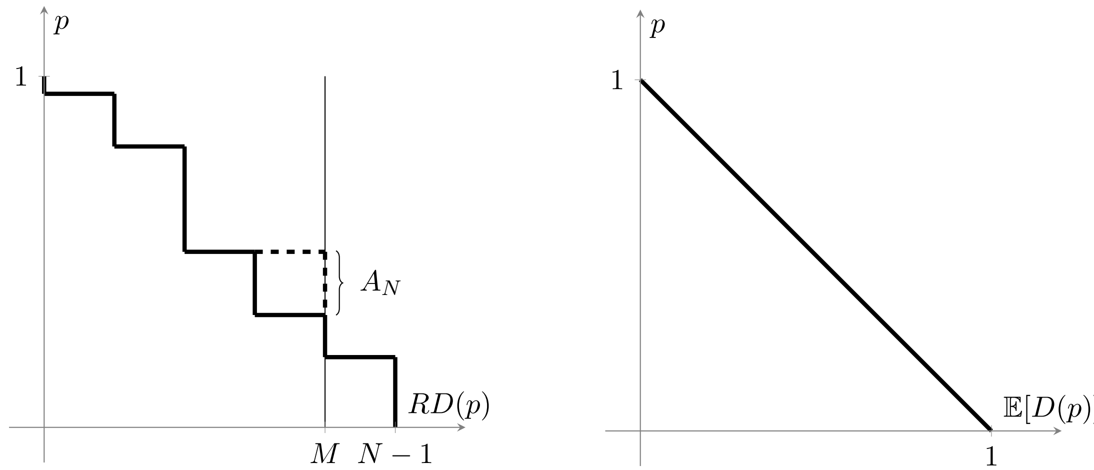
Figure 1: Demand functions for Example 1.1

Here, a small change in one agent's report can substantially impact prices, even when that agent is small compared to the economy's size. The primary reason for this is that the residual demand curve, $RD(p)$ (the sum of the demand curves of buyers 1 through $N-1$), is flat near the equilibrium price (even in the limit as $N \to \infty$), as Figure 1 illustrates. This enables a small change in one agent's reported demand to move the intersection with the supply curve a relatively large distance in price space.

Example 1.2. Now consider an economy with N buyers and a single good with supply $M < N$. Buyer $n \in \{1, ..., N\}$ has unit demand for the good with value a_n, where a_n is drawn independently and uniformly on $[0, 1]$. This example considers the *expected* influence any single agent has on Walrasian equilibrium price(s), with the expectation taken over draws of the N agents.

Consider the problem from the perspective of agent N, supposing that all other agents truthfully report their values to the mechanism designer and that the agent is restricted to reporting unit demand. In this scenario, the set of prices that the agent may realize by *some* report—not necessarily an optimal, or even beneficial, one—is $A_N := [a^{(M-1)}, a^{(M)}]$, where $a^{(i)}$ is the i^{th} order statistic of the $N-1$ *other* draws of the valuation distribution. Because the expected spacings of the uniform order statistics is $O(1/N)$, the expected maximum impact of agent 1 on the equilibrium price is also $O(1/N)$. The same logic holds for any valuation distribution that has full support on some interval, with a density uniformly bounded away from zero (that is, $f(x) > c > 0$ for all $x \in \text{supp}(f)$).[5]

[5] See, for example, Satterthwaite and Williams (1989).



(a) Example of a residual demand curve and attainable prices for buyer N

(b) Expected demand curve for an individual agent (equivalently average demand as $N \to \infty$)

Figure 2: Demand functions for Example 1.2

While the residual demand curve has flat segments for finite N, as $N \to \infty$, these flat segments become small compared to the maximum demand of any individual buyer, with the average demand curve approaching the negative-sloping curve illustrated in Figure 2(b). This expected demand curve, which has a slope uniformly bounded away from zero, is *strongly monotone* (formally introduced in Section 3). In Section 4, I show that this property drives the rapid convergence of incentives toward price-taking behavior in this example.

Related literature The motivating application of the perturbation analysis in this paper is to the study of *ex post* incentives in the Walrasian mechanism. Hurwicz (1972) first observed that agents with private information about their preferences may benefit from strategically misreporting demand in order to influence the price vector. This problem extends beyond Walrasian equilibrium: the celebrated theorem of Green and Laffont (1979) implies that no mechanism in the quasilinear domain is strategy-proof, efficient and budget-balanced. In large markets, however, Roberts and Postlewaite (1976) showed that the benefits of misreporting in a Walrasian mechanisms for any individual agent tends to zero provided that the Walrasian equilibrium price correspondence (mapping measures over the function space of possible excess demand functions to prices) is continuous at the limit economy. Jackson (1992) extended this result, showing that the L^∞ distance between true preferences and an optimal report also tends to zero under that condition. He, Miralles, Pycia, and Yan (2018) use a similar condition to establish approximate incentive compatibility in replica economies associated with pseudomarkets à la Hylland and Zeckhauser (1979). However, these papers do not study the *rates* of convergence, making it difficult to assess the likelihood of good reporting incentives in real-world applications. Furthermore, the regularity and continuity conditions used in these results are challenging to apply, as they rely on attributes of the Walrasian equilibrium price correspondence rather than underlying properties of the agents' preferences.

Rates of convergence of *ex post* incentives have been studied in several specific models, including the unit-demand double auction of Satterthwaite and Williams (1989) and linear-quadratic finance models surveyed by Rostek and Yoon (2020). Satterthwaite and Williams show that, provided values and costs are drawn i.i.d. from full-support distributions with a lower-bounded density, the maximum gain from misreporting is $O(1/N)$ and the distance between true and optimal reports is $O(1/N)$. Like this paper, the finance literature surveyed by Rostek and Yoon, specializing in linear-quadratic preferences, emphasizes the relationship between the slope of aggregate demand and incentives for price-taking behavior in Bayes-Nash equilibrium. This paper identifies the general property of the demand curve that drives the incentive results in these specific models.

Azevedo and Budish (2019) show that all envy-free mechanisms, including Walrasian mechanisms, are "strategy-proof in the large", meaning that the expected benefit to a single agent of misreporting in response to any full-support i.i.d. distribution of opponent reports (from a finite type space) is $O(1/N^{\frac{1}{2}-\varepsilon})$ for any $\varepsilon > 0$. This paper's result is, in a sense, both stronger and weaker than Azevedo and Budish's. It is weaker in one sense because this paper focuses on the Walrasian mechanism and require strong monotonicity. It is stronger in another sense, because the rate of convergence is faster, both ex post and interim incentives are characterized, and the finite type space assumption is not needed.

Al-Najjar and Smorodinsky (2007) adopt an alternative approach to studying the influence of strategic behavior on market mechanisms, focusing on the Bayes-Nash equilibria (BNE) of the associated reporting game. Al-Najjar and Smorodinsky show that for any level of approximation there exists a sufficiently large $\overline{N}$ such that the outcome associated with any BNE of a competitive mechanism with at least $\overline{N}$ participants is approximately efficient. Unlike this paper, their approach does not characterize the ability of an agent to influence on prices, rather the *number* of agents who can influence prices. Moreover, Al-Najjar and Smorodinsky (2007) require a finite type space and a small probability that agents are not strategic.

Organization Section 2 introduces the model and a definition of perturbation-proofness. Section 3 introduces strong monotonicity of demand and establishes perturbation-proofness in large markets where *all* agents have strongly monotone demand (Theorem 1) and in replica economies of markets with strongly monotone demand (Theorem 2). Section 4 studies an independent private value model and characterizes perturbation-proofness under strong monotonicity of expected demand. Finally, Section 5 applies these results to markets with indivisibilities.

2 Exchange economies and perturbations

A market contains L types of consumable goods and a numeraire, money.

There is a finite set of buyers $\mathscr{N}$, with $|\mathscr{N}| = N$. Each buyer $n \in \mathscr{N}$ chooses a bundle $x_n = (x_n^1, \cdots, x_n^L)$ from a convex, compact *consumption possibility set* $X_n \subseteq \mathbb{R}_+^L$, with $0 \in X_n$.

Each buyer has quasilinear preferences over commodity bundles in X_n with a *valuation function* $v_n : X_n \to \mathbb{R}$. Buyer n's *utility* from allocation x_n given payment t is $U_n(x_n, t) := v_n(x_n) - t$. The valuation functions are elements of a function space $\mathscr{V}$, such that each $v_n \in \mathscr{V}$ is monotone, concave[6] and satisfies the normalization $v_n(0) = 0$.

There is an exogenous *supply vector* $s \in \mathbb{R}_{++}^L$ of consumable goods.[7] Buyers are unconstrained in their spending of money. Together, $\mathscr{E} = \langle \mathscr{N}, s \rangle$ is a *market.*

Efficiency, equilibrium and mechanism design An *allocation* $\mathbf{x} = (x_n)_{n \in \mathscr{N}}$ is an assignment of consumption bundles $x_n \in X_n$ to each buyer $n \in \mathscr{N}$. Allocation $\mathbf{x}$ is *feasible* in $\mathscr{E}$ if $\sum_{n \in \mathscr{N}} x_n \leq s$. The set of all feasible allocations for $\mathscr{E}$ is denoted $\mathscr{X}$.

The *surplus* of allocation $\mathbf{x}$ is $\mathscr{S}(\mathbf{x}) := \sum_{n \in \mathscr{N}} v_n(x_n)$. An *efficient allocation* for $\mathscr{E}$ is a feasible allocation $\mathbf{x} \in \mathscr{X}$ solving the surplus maximization problem $\max_{\mathbf{x} \in \mathscr{X}} \mathscr{S}(\mathbf{x})$.

Let $p \in \mathbb{R}_+^L$ be a price vector and $D_n : \mathbb{R}_+^L \rightrightarrows X_n$ the *demand correspondence* of buyer n, defined to be the set of maximizers of $U_n(x, p \cdot x)$. Throughout, I maintain the assumption that, for all $n \in \mathscr{N}$, $D_n(p) = \{0\}$ for prices p outside of a compact set $\mathscr{P} \subseteq \mathbb{R}_+^L$. The indirect utility function $u_n : \mathbb{R}_+^L \to \mathbb{R}$ is $u_n(p) := \max_{x \in X_n} U_n(x, p \cdot x)$. The indirect utility function is convex and related to the demand correspondence by the identity $\partial u_n(p) = -D_n(p)$.

A *Walrasian equilibrium* of $\mathscr{E}$ is an allocation $\mathbf{x} \in \mathscr{X}$ and a price vector $p \in \mathbb{R}_+^L$ such that:

(a) markets clear—that is, $\sum_{n \in \mathscr{N}} x_n \leq s$, and

(b) unwanted goods have zero price—that is, $p^\ell(\sum_{n \in \mathscr{N}} x_n^\ell - s^\ell) = 0$ for each ℓ, and

[6] This assumption primarily ensures the existence of Walrasian equilibria. For analysis of linear pricing mechanisms, including incentives, without the assumption of concavity, see Milgrom and Watt (2021).

[7] In Appendix C, I discuss extensions of the results to settings where supply decisions are made by participants in the mechanism.

(c) assignments are demanded given the price vector—that is, $x_n \in D_n(p)$ for each $n \in \mathscr{N}$.

Let $W(\mathscr{E})$ denote the set of Walrasian equilibrium price vectors of $\mathscr{E}$, which is nonempty under the assumptions on preferences (see Appendix B). By the first welfare theorem, Walrasian equilibrium allocations are always efficient.

Buyers report their preferences to a mechanism designer who determines an outcome and transfers. By the revelation principle, it suffices to study mechanisms where buyers report their valuation functions v_n to the market designer.[8] In the *Walrasian mechanism*, the mechanism designer (or Walrasian auctioneer) determines[9] and implements the Walrasian equilibrium prices and allocations based on the reported valuation functions, with some pre-determined decision rule if the Walrasian equilibrium is not unique. This decision rule plays no key role in this paper.

Perturbations and perturbation-proofness This paper examines small but *finite* perturbations of markets, distinguishing this analysis from the study of shadow prices, relevant only for infinitesimal perturbations.

Definition 2.1. Let $\mathscr{E} = \langle \mathscr{N}, s \rangle$ denote the *original* or *unperturbed market*. A *perturbation* is a vector $\delta s \in \mathbb{R}^L$ such that $s + \delta s \geq 0$. The *perturbed market* is denoted $\mathscr{E}' = \langle \mathscr{N}, s + \delta s \rangle$.

I focus on the influence of perturbations as the market grows large, in the following sense.

Definition 2.2. A *nested market sequence* is a sequence of markets $(\mathscr{E}_t)_{t \in \mathbb{N}}$ where $\mathscr{E}_t = \langle \mathscr{N}_t, s_t \rangle$, and $\mathscr{N}_{t+1} \supseteq \mathscr{N}_t$ for all $t \in \mathbb{N}$ with $N_t \to \infty$.

Definition 2.3. A nested market sequence $(\mathscr{E}_t)_{t \in \mathbb{N}}$ is $O(f(t))$*-perturbation-proof* if for all sequences of perturbations $(\delta s_t)_{t \in \mathbb{N}}$ with size $\|\delta s_t\| \leq O(1)$, $\|p_t - p'_t\| \leq O(f(t))$ for any $p_t \in W(\mathscr{E}_t)$ and $p'_t \in W(\mathscr{E}'_t)$.

Perturbation-proofness requires small perturbations of markets in a nested market sequence to lead to small changes in price. The definition implicitly requires that the set of Walrasian equilibrium prices $W(\mathscr{E}_t)$ is small, with diameter no larger than $O(f(t))$, since $\delta s_t \equiv 0$ is a valid perturbation.

[8] The question of how bidders communicate these potentially complicated objects to the mechanism designer is beyond this paper's scope. The design of bidding languages to report complex preferences has been the subject of substantial study, including by Milgrom (2009), Bichler, Goeree, Mayer, and Shabalin (2014), and Bichler, Milgrom, and Schwarz (2020).

[9] This implicitly assumes that the market designer can efficiently and exactly compute Walrasian equilibrium, which is generally a non-trivial assumption given that Walrasian equilibrium computation problem is PPAD-complete. However, Appendix D shows that Walrasian equilibrium *can* be approximated efficiently in the case of strongly monotone demand using tâtonnement (gradient) methods.

This paper focuses $O(1/N_t)$-perturbation-proofness (or for technical reasons in the independent private values model of Section 4, the very close rate of $O(1/N_t^{1-\varepsilon})$ for all $\varepsilon > 0$). Other than in the precise statements of theorems, "perturbation-proofness" (with the big O notation omitted) will refer to these rates. However, the definition above is general and permits faster or slower rates, as discussed in Appendix A, and rates depending on s_t or δs_t as well as N_t.

At first glance, the definition of a perturbation appears narrow, allowing only changes in supply. However, perturbation-proofness implies robustness—at the same rate—to two other significant changes to the economy.

Proposition 1. *Suppose that $(\mathscr{E}_t)_{t\in\mathbb{N}}$ is $O(f(t))$-perturbation-proof, and consider the following changes:*

(a) Misreporting*: Let n be a buyer in $\mathscr{N}_t$ for each $t \geq T$, and obtain $\mathscr{E}'_t$ by replacing v_n by some $v'_t \in \mathscr{V}$ for $t \geq T$.*

(b) Addition of buyers*: Suppose $\mathscr{N}_0$, a finite subset of buyers with valuations drawn from $\mathscr{V}$, is added to each $\mathscr{E}_t$ to obtain $\mathscr{E}'_t$.*

In both cases, $\|p_t - p'_t\| \leq O(f(t))$ for all $p_t \in W(\mathscr{E}_t)$ and $p'_t \in W(\mathscr{E}'_t)$.

Note that the problem of removing agents can be obtained by swapping the role of $\mathscr{E}_t$ and $\mathscr{E}'_t$ in (b) and the problem of misreporting by a bounded number of agents (that is, $O(1)$ in N_t) by repeated application of (a).

Proof. For brevity, I omit the t index. The necessary conditions for Walrasian equilibrium of the original economy is $s \in D(p)$. For (a), suppose that agent n_0 receives allocation x_0 under truthful reporting and obtains allocation $\tilde{x}$ under a misreport inducing Walrasian equilibrium price $\tilde{p}$. Buyer n_0's announcement must satisfy $s \in \sum_{n\in\mathscr{N}\setminus\{n_0\}} D_n(\tilde{p}) + \tilde{x}$. But this is the same as the necessary conditions for equilibrium of the problem $s + x_0 - \tilde{x} \in D(p)$, which corresponds to a perturbation of $\mathscr{E}$ by $\delta s = x_0 - \tilde{x}$, which is $O(1)$ in N_t since X_n is bounded. Thus the effect of misreporting on prices may be thought of as a perturbation in the sense of Definition 2.1. The effect of adding buyers is similar; the equivalent perturbation is $-\sum_{n\in\mathscr{N}_0} D_n(\tilde{p})$ where $\tilde{p}$ is the induced price in the perturbed economy, which is $O(1)$ in N_t since N_0 and each X_n are bounded.[10]

[10]Morover, the same idea works for the addition (or subtraction) of any Lipschitz convex function to the dual objective of the efficient allocation problem (discussed in Appendix B). Such functions have bounded subdifferentials; thus, the effect of their addition on the necessary and sufficient conditions of the dual problem are equivalent to $O(1)$ changes in the supply vector. Each perturbation discussed above—supply vector changes, misreporting, and the addition of agents—may be interpreted as Lipschitz perturbations of the dual objective.

□

Finally, note the relationship between perturbation-proofness and approximate incentive compatibility of the Walrasian mechanism.

Definition 2.4. A mechanism is $O(f(t))$-*ex post incentive compatible (EPIC)* on $(\mathscr{E}_t)_{t\in\mathbb{N}}$ if truthful reporting is an $O(f(t))$-ex post Nash equilibrium. That is, the payoff to a buyer n from a unilateral misreport is at most $O(f(t))$ greater than the payoff from truthful reporting.

Proposition 2. *If $(\mathscr{E}_t)_{t\in\mathbb{N}}$ is $O(f(t))$-perturbation-proof, then the Walrasian mechanism is $O(f(t))$-EPIC on $(\mathscr{E}_t)_{t\in\mathbb{N}}$.*

Proof. Since $(\mathscr{E}_t)_{t\in\mathbb{N}}$ is perturbation-proof, any misreport by a buyer, including the optimal one, results in a price p'_t satisfying $\|p'_t - p_t\| \leq O(f(t))$, where $p_t \in W(\mathscr{E}_t)$. The buyer's utility from the misreport does not $u_n(p'_t)$, while the buyer's utility from truthful reporting is exactly $u_n(p_t)$, so the gain from misreporting is bounded above by $u_n(p'_t) - u_n(p_t)$. Since u_n is Lipschitz, this expression is at most a constant multiple of $\|p'_t - p_t\|$ which is $O(f(t))$.[11] □

3 Strong monotonicity and complete information exchange economies

In this section, I introduce a condition on demand resulting in perturbation-proofness of exchange economies.

Definition 3.1. Buyer n's demand correspondence $D_n : \mathbb{R}^L_+ \rightrightarrows X_n$ is *strongly monotone* if there exists some $m > 0$ such that for all prices $p, p' \in \mathbb{R}^L_+$ such that $D_n(p) \neq \{0\}$ and $D_n(p') \neq \{0\}$,

$$(d' - d) \cdot (p - p') \geq m\|p - p'\|^2, \text{ for all } d \in D_n(p) \text{ and } d' \in D_n(p'). \quad \text{(SM)}$$

Definition 3.1 adapts the concept of strong monotonicity, developed in the mathematical theory of optimization, to the economic context. This adaptation differs from the standard definition (discussed further in Appendix A) in two key ways: first, a sign change reflecting the fact that D_n is the *negative* subdifferential of the indirect utility function, and second, the requirement that (SM) apply only at prices p, p' where demand is nonzero. Without this latter modification, the law of

[11]The Lipschitz property of u_n follows since the Lipschitz constant of a proper, convex function is the magnitude of the largest selection from a subderivative of that function (see Theorem 9.13 in Rockafellar and Wets (2009)). For u_n, this is a demand bundle, of bounded magnitude by the assumption that X_n is compact.

demand and nonnegativity of demand would imply that if demand at price p is zero, the inequality (SM) could not be satisfied for prices $p' = \alpha p$ with $\alpha > 1$.

Note the resemblance of (SM) to the law of demand, which obtains when the right-hand side of (SM) is set to zero. However, whereas the law of demand is satisfied for *all* demand correspondences, not all demand correspondences are strongly monotone.

Markets with strongly monotone agents I first analyze markets where *all* agents have strongly monotone demand. This assumption is strong—it is not satisfied, for example, in the unit demand valuations of Example 1.2—but this analysis provides intuition for other results.

Theorem 1. *Suppose each $v_n \in \mathscr{V}$ has a strongly monotone demand correspondence with constant $m > 0$. Then any nested market sequence $(\mathscr{E}_t)_{t\in\mathbb{N}}$ is $O(1/N_t^a)$-perturbation-proof where N_t^a is the number of buyers with $D_n(p_t) \neq \{0\}$ for $p_t = W(\mathscr{E}_t)$ and $D_n(p_t') \neq \{0\}$ and $p_t' = W(\mathscr{E}_t')$.*

Note that $W(\mathscr{E}_t)$ and $W(\mathscr{E}_t')$ are singleton under the assumption that all agents have strongly monotone demand as strong monotonicity implies that inverse demand is single-valued (except at zero).[12]

With one type of consumable good, the strong monotonicity assumption on demand curves in $\mathscr{V}$ implies that the slope of the market demand curve at any price, including the equilibrium price, grows proportionally to the number of agents with nonzero demand at that price. As the demand curve becomes increasingly steep at the equilibrium price, small perturbations of the supply vector have a progressively smaller effect on the resulting price.[13]

Proof. By assumption, for each active buyer n at price,s p_t, p_t' and for any $d_n \in D_n(p_t)$ and $d_n' \in D_n(p_t')$

$$(d_n' - d_n) \cdot (p_t - p_t') \geq m\|p_t - p_t'\|^2, \tag{1}$$

so that

$$\sum_{n\in\mathscr{N}_t} (d_n' - d_n) \cdot (p_t - p_t') \geq N_t^a m\|p_t - p_t'\|^2. \tag{2}$$

That is, the aggregate demand satisfies (SM) with constant $N_t^a m$. Adding $(s - \sum_{n\in\mathscr{N}_t} d_n) \cdot p_t' + (s' - \sum_{n\in\mathscr{N}_t} d_n') \cdot p_t$, which is nonnegative by feasibility and the nonnegativity of price, to (2) and

[12] Alternatively, using the notions developed in Appendix A and Appendix B, the dual objective is strongly convex—and thus strictly convex—so the equilibrium price is always unique.

[13] In Appendix D, I show that this same logic also implies that markets in which all agents have strongly monotone demand are tâtonnement-stable, with a fast (subpolynomial) rate of convergence of tâtonnement.

exploiting the complementary slackness conditions for p, p' implies

$$(s' - s) \cdot (p_t - p'_t) \geq N_t^a m \|p_t - p'_t\|^2.$$

Since $s' - s = \delta s$, by the Cauchy-Schwarz inequality,

$$\|\delta s\| \left\|p_t - p'_t\right\| \geq N_t^a m \left\|p_t - p'_t\right\|^2,$$

or on re-arranging,

$$\|p_t - p'_t\| \leq \frac{\|\delta s\|}{m N_t^a} \leq O\left(\frac{1}{N_t^a}\right).$$

□

Necessary conditions and replica economies In the remainder of this section and Section 4, I weaken the assumption in Theorem 1 that *each* agent has strongly monotone demand. This requires more structure on the sequence of economies (as in Theorem 2) or weaker conclusions (in Section 4, I settle for probabilistic results).

An alternative, if imprecise, interpretation of the proof of Theorem 1 motivates my approach. With one consumable good, when all agents have strongly monotone demand functions, the *average* (per-capita) demand curve (averaged over the number of active buyers at price p_t) is downward sloping. The market-clearing condition of the Walrasian mechanism must hold for the per-capita economy with per-capita supply. On the other hand, the per-capita perturbation diminishes at a rate inversely proportional to the number of active buyers at price p_t. So, if the average demand curve is bounded away from zero at p_t, the effect of a perturbation on prices diminishes at the same rate as the per-capita size of the perturbation.

This suggests the importance of the *average* demand correspondence for the analysis of perturbations. For a given average demand correspondence, the simplest associated sequence of economies is the replica economies of Debreu and Scarf (1963).

Definition 3.2. Let $\mathscr{N}$ be a set of buyers and define $\mathscr{N}_k$, its $k-$*fold replica*, as the set of kN buyers such that for each $n \in \mathscr{N}$, there are k buyers in $\mathscr{N}_k$ with the same preferences as n. The $k-$fold replica of a *base economy* $\mathscr{E} = \langle N, s \rangle$ is $\mathscr{E}_k = \langle \mathscr{N}_k, ks \rangle$.

In replica economies, the average demand correspondence is constant with respect to the number of replicas. In this setting, Theorem 2 states that strong monotonicity of the average demand correspondence is a necessary and sufficient condition for the conclusions of Theorem 1 to hold for all possible supply vectors s.

Theorem 2. *Let N be a market with total demand correspondence $D = \sum_{n \in \mathscr{N}} D_n$, and let $\mathscr{N}_k$ be its $k-$fold replica. Then $\mathscr{E}_k = \langle \mathscr{N}_k, ks \rangle$ is $O(1/N_k)$-perturbation-proof for all base economy supply vectors s if and only if D is strongly monotone.*

4 Markets with private valuations

For the remainder of this paper, I analyze an independent private values model of a Walrasian economy. Instead of drawing a single value parameter according to a common prior distribution, agents draw preferences from an abstract (potentially infinite-dimensional) function space.

Definition 4.1. Let $\mathscr{V}$, equipped with an appropriate σ-algebra, be a measurable space of valuation functions, [14] and let ν be a distribution over $\mathscr{V}$, which is common knowledge. A market $\mathscr{E} = \langle N, s \rangle$ has *independent private values (IPV)* if the valuation function of each buyer in $\mathscr{N}$ is drawn identically and independently from ν.

In this section, I make one key assumption on the set of buyer types.

Assumption 1. *Given distribution ν on $\mathscr{V}$, there exists a compact set $\mathscr{P} \subseteq \mathbb{R}^L_+$ such that $D_n(p) = \{0\}$ ν-almost surely for $p \notin \mathscr{P}$.*

Assumption 1 is embedded in many auction models and will be required for the main results in this section.[15]

I now introduce the notion of strong monotonicity in this environment.

Definition 4.2. Given distribution ν on $\mathscr{V}$, define the *expected indirect utility function* pointwise for $p \in \mathscr{P}$ by

$$\mathbb{E}_\nu[u(p)] = \int_{\mathscr{V}} u_n(p) d\nu(u_n).$$

[14]For example, by a result of Aumann (1963), $\mathscr{V}$ could be taken as the set of bounded, continuous functions on a compact subset of $\mathbb{R}^L_+$, or the set of bounded functions with discontinuities of the first kind, or, more generally, any subset of a Baire class. In particular, the set of valuation functions that are monotone, concave and normalized in accordance with previous assumptions is admissible.

[15]It is simple to modify results to require only that the equilibrium price belongs to a compact set $\mathscr{P}$ almost surely, but the formulation of Assumption 1 is sufficient because it does not require knowledge of supply vector s.

The *expected demand correspondence* is defined by [16]

$$\mathbb{E}_\nu[D(p)] = -\partial \mathbb{E}_\nu[u(p)].$$

Say that ν on $\mathscr{V}$ has *strong monotonicity in expectation* if $\mathbb{E}_\nu[D(p)]$ is a strongly monotone demand correspondence in the sense of Definition 3.1.

Note that strong monotonicity in expectation does not require that the individual agents' demands are strongly monotone. For example, in Example 1.2, the individual demand correspondences are step functions (not strongly monotone), while the expected demand function is strongly monotone.

Perturbation-proofness in probability I now introduce a probabilistic notion of perturbation-proofness that is appropriate in economic settings with private valuations.

Definition 4.3. Let $(\mathscr{E}_t)_{t\in\mathbb{N}}$ be a nested sequence of markets with private valuations drawn i.i.d. from distribution ν on $\mathscr{V}$. The sequence $(\mathscr{E}_t)_{t\in\mathbb{N}}$ is $O(f(t))$*-perturbation-proof with probability* $g(t)$ if for all sequences $(\delta s_t)_{t\in\mathbb{N}}$ of perturbations with size $\|\delta s_t\| \leq O(1)$, $\|p_t - p'_t\| \leq O(f(t))$ for $p_t \in W(\mathscr{E}_t)$ and $p'_t \in W(\mathscr{E}'_t)$, with probability $g(t)$ over draws of $\mathscr{E}_t$.

Strong monotonicity in expectation of μ implies the following.

Theorem 3. *Let $(\mathscr{E}_t)_{t\in\mathbb{N}}$ be a nested sequence of markets with private valuations drawn i.i.d. from distribution ν on $\mathscr{V}$, and suppose that ν on $\mathscr{V}$ has strong monotonicity in expectation. Then for all $\varepsilon > 0$, $(\mathscr{E}_t)_{t\in\mathbb{N}}$ is $O(1/N_t^{1-\varepsilon})$-perturbation-proof with probability $1 - O(1/N_t^{1-\varepsilon})$.*

The proof of Theorem 3, presented in Appendix E, involves establishing the concentration of the empirical average demand correspondence around the expected demand correspondence, via an application of Bernstein's Inequality.

Reporting incentives I now describe the implications of Theorem 3 for reporting incentives in Walrasian mechanisms. These results pertain to incentives under two informational structures, first, *ex post* incentive compatibility as in Definition 2.4 in which agents choose their reports with

[16] The expected demand correspondence can be defined equivalently using the set-valued integral of Aumann (1965). That is, for any fixed p, the probability measure ν induces a probability measure over the sets $D_n(p)$ associated with valuation function $v_n \in \mathscr{V}$. A selection $\xi : \mathscr{V} \to X$ is a single-valued random vector such that $\xi(v_n)$ ν-almost surely belongs to $D_n(p)$ for each $v_n \in \mathscr{V}$. Then $\mathbb{E}_\nu[D_n(p)]$ is defined as $\mathrm{cl}(\{\mathbb{E}_\nu \xi\})$ over integrable selections ξ. A result of Rockafellar and Wets (1982) implies equivalence with Definition 4.2. Moreover, a law of large numbers applies to $\mathbb{E}_\nu[D_n(p)]$ pointwise so that for all $p \in \mathscr{P}$, $d_H(\frac{1}{|N_t|}\sum_{n\in N_t} D_n(p), \mathbb{E}_\nu[D_n(p)]) \to 0$ as $|N_t| \to \infty$, where N_t is a set of agents drawn i.i.d. from ν (Weil, 1982).

common knowledge of all agents' drawn valuations and, second, *interim* incentive compatibility, defined below.

Definition 4.4. Let $(\mathscr{E}_t)_{t\in\mathbb{N}}$ be a nested sequence of markets with private valuations drawn i.i.d. from distribution ν on $\mathscr{V}$. Suppose that each agent chooses its report with knowledge of its own valuation, without knowing the valuations drawn by other agents, and with s_t, ν and the number of agents common knowledge. Then $(\mathscr{E}_t)_{t\in\mathbb{N}}$ is *interim* $O(f(t))$*-incentive compatible* if, for each agent in the mechanism, the expected payoff of the optimal report minus the expected payoff of the truthful report is $O(f(t))$.

The following incentive properties of Walrasian mechanisms follow from Theorem 3.

Theorem 4. *Let $(\mathscr{E}_t)_{t\in\mathbb{N}}$ be a nested sequence of markets with private valuations drawn i.i.d. from distribution ν on $\mathscr{V}$, and suppose that ν on $\mathscr{V}$ has strong monotonicity in expectation. Then for all $\varepsilon > 0$, a Walrasian mechanism on $(\mathscr{E}_t)_{t\in\mathbb{N}}$ is*

(a) ex post $O(1/N_t^{1-\varepsilon})$-incentive compatible with probability $1 - O(1/N_t^{1-\varepsilon})$ over draws of $\mathscr{E}_t$.

(b) interim $O(1/N_t^{1-\varepsilon})$-incentive compatible.

5 Strong monotonicity with indivisible goods

In this section, I specialize my analysis to economies with indivisible goods, so that $X_n \subseteq \mathbb{Z}_+^L$. Economies with indivisibilities are of interest for two key reasons: first, indivisibilities are a natural assumption in many important markets, and second, the change in demand associated with any price change is bounded below by the size of the indivisibility. This allows me to focus, for the purpose of establishing (expected) strong monotonicity, on the prices at which demand changes, rather than also concerning ourselves with the size of these demand changes.

In models with indivisibilities, strong monotonicity of individual demand cannot be observed since prices are a continuous variable while demand can take on only finitely many values. For this reason, I focus on models with incomplete information, as in Section 4. The main goal is to establish conditions under which the expected demand correspondence is strongly monotone. In doing so, I will establish $O(1/N)-$incentive-compatibility for models where rates of convergence have not previously been established. I will also offer an interpretation of the strong monotonicity assumption in models with indivisibilities.

Recall that for economies with one good, unit demand buyers and uncertainty in valuations (as in Example 1.2), a sufficient condition for the expected maximum influence on price by any single buyer to be $O(1/N)$ is that a_n be drawn from a distribution with full support on an interval in $\mathbb{R}$ with density bounded below by $\alpha > 0$ (Rustichini, Satterthwaite, and Williams, 1994). This condition guarantees strong monotonicity of the expected demand since, for $p' > p$, the change in demand grows at least linearly with $p' - p$, that is

$$\mathbb{E}[d(p') - d(p)] \geq \int_{p}^{p'} \alpha d\tilde{p} = \alpha(p' - p).$$

This implies the required inequality, $\mathbb{E}[(d(p') - d(p))(p' - p)] \geq \alpha(p' - p)^2$.

It should be clear that the unit demand structure is not necessary for this result: all that is required is that, for any price p, there is a positive probability to draw marginal buyers and non-buyers of the good (that is, buyer types who would reduce demand in response to a price increase and types who would increase demand in response to a price decrease) and a condition that corresponds to a lower-bounded density. With more goods, one must also consider the many directions in which price changes can occur at any given price. I formalize this intuition in the following proposition.

Proposition 3 (Expected strong monotonicity for multiple indivisible goods). *Let $\mathscr{P}$ be a compact, convex subset of $\mathbb{R}^L_+$. Suppose there exists some $\alpha > 0$ such that for all $p, p' \in \mathscr{P}$ with $p \neq p'$, $\Pr_\nu[D_v(p) \neq D_v(p')] \geq \min\{\alpha\|p' - p\|, 1\}$ for some $\alpha > 0$. Then the expected demand correspondence associated with ν is strongly monotone.*

This condition may be interpreted in terms of two natural assumptions for economic models with indivisibilities. First, *uncertainty* about where (in price space) a demand change occurs: there must be some probability that demand changes (for a type drawn from ν) associated with *any* price change, and larger price changes must lead to a proportionately larger probability that demand changes. Second, and more subtly, the condition reflects *variety* in the preferences. To see this, fix a price p and consider small price changes in each of the coordinate directions from p. For each such price change, there must be some probability that demand changes and by the law of demand, these demand changes must be non-orthogonal to the price change.[17] Taking the limit as the price

[17] This is a slightly stronger version of the usual law of demand that applies when demand strictly changes. If $x \in D(p)$ and $x \notin D(p') \ni x'$ then $u(x) - p \cdot x \geq u(x') - p \cdot x'$ and $u(x') - p' \cdot x' > u(x) - p' \cdot x$. Adding and rearranging gives $(p' - p) \cdot (x - x') > 0$.

changes approach zero,[18] it is possible that demand changes at p in non-orthogonal directions. For example, at price p, some types in the support of ν might be indifferent to buying or not buying good x, while other types are indifferent to buying or not buying good y. In other words, each price p must be a kind of "marginal price" for different demand changes for various agent types. This interpretation is reminiscent of Hildenbrand's (1994) argument that the law of demand reflects primarily "heterogeneity" of the population of households.[19]

Finally, I show by example how Proposition 3 can be used to establish perturbation-proofness (and therefore incentive results) in a novel economic environment. This is suggestive of many new economic environments where approximate incentive-compatibility results may be obtained which have not previously been established.

Example 5.1 (Complementarities)**.** Suppose there are two goods x and y, with $(x, y) \in \{0, 1\}^2$. Buyer n has the following valuation function for the goods

$$v_n(x, y) = a_n x + b_n y + c_n xy,$$

where each of a_n, b_n and c_n are strictly positive real numbers. The demand for such a buyer is illustrated in Figure 3.

Suppose that a_n and b_n are drawn independently from full support distributions on $[0, 1]$ and c_n is drawn from a full support distribution on $[0, 1 - a_n \wedge 1 - b_n]$, all with densities bounded below.

Let $\mathscr{P} = [0, 1]$ and prices $p, p' \in \mathscr{P}$ be given (without loss of generality, suppose $p_x \leq p'_x$). There are two possibilities, either

(a) $p_x \neq p'_x$. In this case, buyers with $a_n \leq p_x$, $b_n \leq p'_y$ and $c_n \in [p'_x - a_n, 1 - a_n]$ change demand by a multiple of $(1, 0)$ as p changes to p'.

(b) $p_x = p'_x$. In this case, buyers with $b_n < p_y$ and $b_n + c_n < p'_y$ with $a_n > p_x$ change demand by a multiple of $(0, 1)$ as p changes to p'.

In each case, the probability of drawing such agents grows in $\|p - p'\|$. Note in both cases that there are other buyers who experience demand changes for this price change but a lower bound on

[18] Assuming a sense of continuity of $\mathscr{V}$: namely that if there are demand correspondences $d_n \in \mathscr{V}$ approaching d in the L^∞-norm, then $d \in \mathscr{V}$. Otherwise this analysis applies generically.

[19] Hildenbrand (1994) argues that the law of demand may be derived at the market level from assumptions on the heterogeneity of household, rather than primarily reflecting the rationality of individual agents, as in the classical approach. While I maintain classical rationality assumptions, the forces driving his law of demand and strong monotonicity of expected demand are similar.

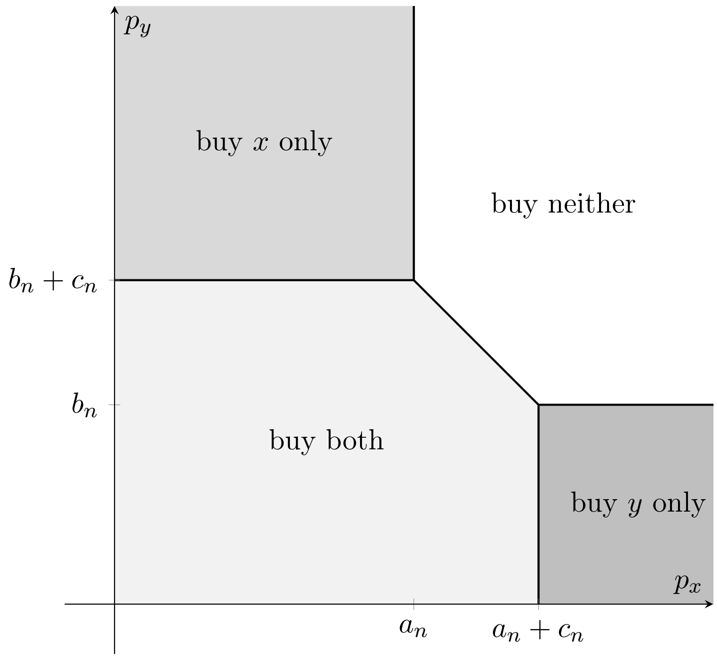


Figure 3: Demands for Example 5.1

the probability of demand changes associated with the price change is sufficient. This implies the required condition in Proposition 3 and so markets consisting of buyers with such complementarities are perturbation-proof, and so the Walrasian mechanism is ex post $O(1/N^{1-\varepsilon})$-incentive compatible with high probability.

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A Strong monotonicity and related notions

I first introduce a stronger notion of convexity that is used in the perturbation analysis of convex programs. Strong convexity is used routinely in the analysis of convex optimization problems, see, for example, Boyd and Vandenberghe (2004). In the following definitions, K is a compact, convex subset of $\mathbb{R}^N$ and $f : K \to \mathbb{R}$ is a convex function defined on K.

Definition A.1 (Order γ-strong convexity)**.** For $\gamma > 0$, the function f is *order γ-strongly convex* with constant $m > 0$ if

$$f(y) \geq f(x) + s_x \cdot (y - x) + \frac{m}{2}\|y - x\|^\gamma \text{ for all } x, y \in K \text{ and } s_x \in \partial f(x).$$

If $\gamma = 2$, simply say that f is strongly convex with constant m.

Note that by replacing m with zero in Definition A.1 recovers a definition of convexity of function f so that Definition A.1 is a stronger condition than convexity. Informally, a function is order γ-strongly convex if it is possible to fit an order γ polynomial between the function and all of its tangent planes.

In the same way that the convexity of function f is equivalent to the monotonicity of its subdifferential ∂f, the (order γ-)strong convexity of f is equivalent to the (order γ-)*strong monotonicity* of ∂f, as defined below.

Definition A.2 (Order γ-strong monotonicity)**.** Let $s : K \rightrightarrows \mathbb{R}$ be a nonempty-valued correspondence defined on K. For $\gamma > 0$, correspondence s is *order $\gamma-$strongly monotone* with constant $m' > 0$ if

$$(s_y - s_x) \cdot (y - x) \geq m' \|y - x\|^\gamma, \text{ for all } x, y \in K \text{ and } s_x \in s(x), s_y \in s(y).$$

For $\gamma = 2$, just say that s is strongly monotone with constant m'.

Note that by replacing m' with zero in Definition A.2 recovers the usual definition of a monotone correspondence.

Proposition 4. *Let $f : K \to \mathbb{R}$ be a convex function and $\partial f : K \rightrightarrows \mathbb{R}$ be its subdifferential mapping.*

(a) If f is order $\gamma-$strongly convex with constant $m > 0$, then ∂f is order $\gamma-$strongly monotone with constant m.

(b) If ∂f is order $\gamma-$strongly monotone with constant $m' > 0$, then f is order $\gamma-$strongly convex with constant $2m'/\gamma$.

Proof. First, suppose f is order $\gamma-$strongly convex. Then,

$$f(y) \geq f(x) + s_x \cdot (y - x) + \frac{m}{2}\|y - x\|^\gamma$$
$$f(x) \geq f(y) + s_y \cdot (x - y) + \frac{m}{2}\|y - x\|^\gamma.$$

Adding these expressions and reorganizing obtains $(s_y - s_x) \cdot (y - x) \geq m\|y - x\|^\gamma$.

For the converse, define $\phi(\lambda) = f(x + \lambda(y - x))$ and $x_\lambda = x + \lambda(y - x)$. Then since $\frac{d\phi}{d\lambda}$ exists almost everywhere and equals $s_\lambda \cdot (y - x)$ for $s_\lambda \in \partial f(x_\lambda)$,by the fundamental theorem of calculus,

$$f(y) - f(x) = \phi(1) - \phi(0) = \int_0^1 s_\lambda \cdot (y - x) d\lambda.$$

By assumption, $(s_\lambda - s_x)\cdot(x_\lambda - x) \geq m\|x_\lambda - x\|^\gamma$. This implies that $\lambda(s_\lambda - s_x)\cdot(y-x) \geq m\lambda^\gamma\|y-x\|^\gamma$. Substituting into the expression above gives

$$f(y) - f(x) \geq s_x \cdot (y-x) + \int_0^1 m\lambda^{\gamma-1}\|y-x\|^\gamma d\lambda = s_x \cdot (y-x) + \frac{m}{\gamma}\|y-x\|^\gamma.$$

□

There are several other well-known characterizations of (order 2-)strong convexity (see Boyd and Vandenberghe (2004)). Function f is strongly convex if and only if:

(a) $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) - \frac{m}{2}\lambda(1-\lambda)\|y-x\|^2$ for all $x, y \in K$ and $\lambda \in [0,1]$.

(b) the function $g : K \to \mathbb{R}$ defined by $g(x) = f(x) - \frac{m}{2}\|x\|^2$ is convex.

Wherever f is twice continuously differentiable, strong convexity requires that $D^2 f(x) - mI$ is positive semi-definite.

Strong convexity also has a dual formulation. Recall that the Fenchel dual of a proper convex function $f : K \to \mathbb{R} \cup \{\pm\infty\}$ defined on $K \subseteq \mathbb{R}^N$ is the function $f^* : \mathbb{R}^N \to \mathbb{R} \cup \{\pm\infty\}$ satisfying $f^*(x^*) = \sup_{x\in K} x^* \cdot x - f(x)$. The following dual characterization of strong convexity is known (see Borwein and Vanderwerff (2010)).

Proposition 5. *A proper convex function $f : K \to \mathbb{R}$ is strongly convex with constant m if and only if the Fenchel dual $f^* : \mathbb{R}^N \to \mathbb{R}$ is* strongly smooth*, that is,*

$$f^*(y^*) \leq f^*(x^*) + s \cdot (y^* - x^*) + \frac{1}{2m}\|y^* - x^*\|^2, \text{ for all } x^*, y^* \in \mathbb{R}^N \text{ and } s \in \partial f^*(x^*).$$

Equivalently, for all $x^*, y^* \in \mathbb{R}^N$, $s_x \in \partial f^*(x^*)$ and $s_y \in \partial f^*(y^*)$, $(s_y - s_x)\cdot(y - x) \leq \frac{1}{m}\|y^* - x^*\|^2$. This latter condition implies the Lipschitz-continuity of ∇f^* wherever it exists.

The following generalization of Theorem 1 applies for order γ-strong monotonicity.

Proposition 6. *Consider a nested market sequence $(\mathscr{E}_t)_{t\in\mathbb{N}}$ in which all agents have order γ-strongly convex preferences with constant $m > 0$. Then $(\mathscr{E}_t)_{t\in\mathbb{N}}$ is $O(1/N_t^{a\,\frac{1}{\gamma-1}})$-perturbation-proof where N_t^a is the number of buyers who are active at some prices $p_t \in W(\mathscr{E}_t)$ and $p_t' \in W(\mathscr{E}_t')$.*

The proof is almost identical to the proof of Theorem 1 and is omitted. The other main results, Theorem 3, Theorem 4 and the sufficiency direction of Theorem 2, can also be adapted in obvious ways for the alternative assumption of order γ-strong monotonicity.

A notion related to strong convexity that may be used when the set of minimizers of a function is not unique is the following.

Definition A.3 (Growth conditions)**.** Let S be the set of minimizers of f on K, supposed to be non-empty, and let $f_0 = \min_{x \in K} f(x)$. For $\gamma > 0$, the function f satisfies the *order γ-growth condition* if there exists some constant $m > 0$ such that for all $x \in K$,

$$f(x) \geq f_0 + \frac{m}{2}[\text{dist}(x, S)]^{\gamma}. \tag{GC}$$

For $\gamma = 2$, call this the *quadratic growth condition.* If (GC) is satisfied only in some neighborhood of x, then refer to it as the *local* order γ-growth condition at x.

Growth conditions were introduced by Shapiro (1992) and thoroughly studied in Bonnans and Shapiro (2013). Because the zero vector is in the subdifferential of f at any minimizer of f, it is clear that the order $\gamma-$growth condition is a weaker concept than order $\gamma-$strong convexity. Proposition 6 is easily modified to apply to $d_H(P_t, P_t')$ under the assumption that N_t has the local order γ-growth condition at P_t for each t. Note, however, that the price selection rule for the Walrasian mechanism in the case of non-unique prices may now matter, since P_t and/or P_t' may not be $O(f(t))$ even when $d_H(P_t, P_t')$ is $O(f(t))$.

B Welfare theorems and equilibrium formulations

The fundamental theorems of welfare economics, as formalized by Arrow (1951), imply that the set of allocations associated with Walrasian equilibria coincide with the set of efficient allocations, which solve the problem

$$\max_{\mathbf{x} \in \mathscr{X}} \mathcal{S}(\mathbf{x}), \tag{OPT}$$

One way to see this is to consider the Lagrangian $\mathscr{L} : \mathscr{X} \times \mathbb{R}^L_+ \to \mathbb{R}$ associated with (OPT), given by

$$\mathscr{L}(\mathbf{x}, p) = \sum_{n \in N} v_n(x_n) + p \cdot (s - \sum_{n \in N} x_n). \tag{L}$$

Since Slater's constraint qualification[20] is satisfied in (OPT) (because the zero allocation is in the

[20] See, for example, Boyd and Vandenberghe (2004).

relative interior of the constraint space), any saddle point of $\mathscr{L}$, that is, any pair $(\mathbf{x}, p)$ that solves

$$\min_{p \in \mathbb{R}^L_+} \max_{\mathbf{x} \in \mathscr{X}} \mathscr{L}(\mathbf{x}, p), \tag{SP}$$

gives rise to a solution $\mathbf{x}$ to the convex program (OPT). Furthermore, the the values of programs (OPT) and (SP) are the same (this is the complementary slackness condition).

From (SP), any saddle point $(\mathbf{x}, p)$ must satisfy $x_n \in \arg\max_{x \in X_n} v_n(x) - p \cdot x$ for each $n \in N$, which is the individual optimality property of Walrasian equilibrium. So the saddle points of (SP)–Walrasian equilibria–correspond to maximizers of (OPT)–efficient allocations–and *vice versa*. This is a statement of the fundamental welfare theorems for quasilinear economies. Moreover, since the objective in (OPT) is bounded and concave and the set $\mathscr{X}$ is compact, (OPT) has a solution and a Walrasian equilibrium exists.

The *dual problem*,

$$\min_{p \in \mathbb{R}^L_+} p \cdot s + \sum_{n \in N} u_n(p). \tag{D}$$

obtained by reorganizing (SP), plays a major role in the analysis of this paper. An advantage of studying the dual problem is that it is an unconstrained convex program. Writing $U(p) = \sum_{n \in N} u_n(p)$ for the total indirect utility function, the first-order (necessary) conditions of (D) are exactly the market-clearing conditions of the Walrasian equilibrium, $s \in -\partial U(p) = \sum_{n \in N} D_n(p)$.

C Production economies

In this section, I discuss extensions of the results to economic environments in which the supply vector is not fixed and instead determined by agents' production decisions. I consider two structures for such economies: first, two-sided markets in which agents are *either* buyers or sellers, and second, markets in which agents may be buyers *or* sellers of certain types of goods given the prices—I call such agents "traders". Most of the proofs are very similar to the exchange economy proofs of the main paper, and so many details are omitted.

Two-sided markets The buyer side of the economy is as in Section 2. Now add a set of sellers $\mathscr{F}$ of cardinality F. Each seller f can produce bundles Y_f, where Y_f is a convex, compact subset of $\mathbb{R}^L_+$ containing 0 called the production set. Each seller has cost function $c_f : Y_f \to \mathbb{R}$ resulting in profits $\Pi_f(y_f, t) = t - c_f(y_f)$. The cost functions are drawn from function space $\mathscr{C}$, such that each $c_f \in \mathscr{C}$

is monotone, convex and satisfies the normalization $c_f(0) = 0$. The pair $\mathscr{E} = \langle \mathscr{N}, \mathscr{F} \rangle$ is a market.

An allocation $\omega = ((x_n)_{n\in\mathscr{N}}, (y_f)_{f\in\mathscr{F}})$ maximizes the surplus $\mathscr{S}(\omega) = \sum_{n\in\mathscr{N}} v_n(x_n) - \sum_{f\in\mathscr{F}} c_f(y_f)$. Let $S_f(p)$ be the seller's supply correspondence and $\pi_f(p)$ the seller's indirect profit function, the maximizers and value function respectively of $\max_{y\in Y_f} \Pi_f(y, p\cdot y)$, related by $S_f(p) = \partial\pi_f(p)$.

For a fixed market $\mathscr{E}$, let $Z(p) = \sum_{n\in\mathscr{N}} D_n(p) - \sum_{f\in\mathscr{F}} S_f(p)$ be the excess demand correspondence. I assume that for any $M > 0$, $\|z\| < -M$ or $\|z\| > M$ for all $z \in Z(p)$ and p outside of a compact set $\mathscr{P}_M$. This is analogous to the assumption that $D(p) = \{0\}$ outside of $\mathscr{P}$ in the main text.

A Walrasian equilibrium is a pair (ω, p) such that $\sum_{n\in\mathscr{N}} x_n \leq \sum_{f\in\mathscr{F}} y_f$, $p^\ell \left(\sum_{n\in\mathscr{N}} x_n - \sum_{f\in\mathscr{F}} y_f\right) = 0$, $x_n \in D_n(p)$ and $y_f \in S_f(p)$. Walrasian equilibria exist in this setting and are efficient.

Whereas Walrasian equilibria solve $0 \in Z(p)$, a perturbation in this setting is now a solution to $\delta s \in Z(p)$. Definition 2.3 now applies to prices $p_t \in W(\mathscr{E}_t)$ and prices p'_t with $Z_t(p') \ni \delta s_t$. The direct analogies of Proposition 1 and Proposition 2 apply to this altered definition of perturbation-proofness.

Seller f is active at price p if $S_f(p) \neq \{0\}$ and it is not the case that for all $\beta > 1$, $S_f(\beta p) = S_f(p)$. This latter case reflects the possibility that the seller is producing at a boundary of its production possibility set and is needed because upper bounds are not imposed on prices.[21] The seller has *strongly monotone supply* if there is some $m^F > 0$ for which $(y - y') \cdot (p - p') \geq m^F \|p - p'\|^2$ for all active prices p, p' and $y \in S_f(p), y' \in S_f(p')$.

Theorem 1 now applies to economies in which all buyers have strongly monotone demand, all sellers have strongly monotone supply, and N_t^a is the total number of active buyers and sellers at prices $p_t \in W(\mathscr{E}_t)$ and p'_t with $Z(p'_t) = \delta s_t$. The proof is similar, where now the expressions for strong monotonicity of demand and supply are added in order to obtain expressions for excess demand. That is, add

$$(y_f - y'_f) \cdot (p_t - p'_t) \geq m^F \|p_t - p'_t\|^2$$

for all sellers to (2) to obtain

$$\left(\sum_{n\in\mathscr{N}_t} (d'_n - d_n) + \sum_{f\in\mathscr{F}_t} (y_f - y'_f)\right) \cdot (p_t - p'_t) \geq \min\{m^N, m^F\}(N_t^a + F_t^a)\|p_t - p'_t\|^2,$$

with the term in parentheses on the left corresponding to the new definition of perturbation δs_t.

Theorem 2, Theorem 3 and Theorem 4 also apply with the appropriate modifications, where in all cases the proofs are modified to exploit strong monotonicity of the (expected) *excess* demand

[21] A similar condition is not needed in the buyers' case because prices are bounded below by zero. An alternative would be to allow sellers to produce unbounded quantities of goods.

functions, and the (expected) dual objective now taking the form $\sum_{n\in\mathscr{N}} u_n(p) + \sum_{f\in\mathscr{F}} \pi_f(p)$.

Markets with traders I now replace the set of buyers with a set $\mathscr{T}$ of traders of cardinality T. Each trader $n \in \mathscr{T}$ has access to bundles X_n, where X_n is a convex, compact subset of $\mathbb{R}^L$ containing 0, called the netput set. Vectors $x \in X_n$ may be positive in some components and negative in others. Each trader has a net-value function $v_n : X_n \to \mathbb{R}$ which may be positive or negative, monotone, concave and satisfying $v_n(0) = 0$, resulting in payoffs $U_n(x,t) = v_n(x) - t$. The remaining formulation is the same as in the main text, except:

(a) Walrasian equilibrium is now defined as $\sum_{n\in\mathscr{T}} x_n \leq 0$, $p^\ell \sum_{t\in\mathscr{T}} x_t^\ell = 0$, and $x_n \in D_n(p)$ for each $n \in \mathscr{T}$.

(b) an agent is 'active' if it is *not* the case that $D_n(p) = D_n(\beta p)$ for all $\beta > 1$, replacing **??**.

(c) for all $M > 0$, there exists a compact set $\mathscr{P}_M$ such that for all $p \notin \mathscr{P}_M$ and $x \in D_n(p)$, $\|x\| < -M$ or $\|x\| > M$.

D Tâtonnement stability of strongly monotone economies

Recall the continuous-time tâtonnement process in which prices are adjusted in proportion to the excess demand for the relevant good:

$$\frac{dp}{dt} = \alpha[D(p(t)) - s], \text{ with } p(0) = p_0$$

for some adjustment speed $\alpha > 0$ and starting price $p_0 \in \mathscr{P}$. Here, I assume $D(p)$ is single-valued, as is in the case of interest where D is a strongly monotone demand correspondence.

It is well-known that in quasilinear economies (and other economies in which there is a representative consumer) that $\lim_{t\to\infty} p(t)$ is a Walrasian equilibrium price for any starting price p_0.

However, in general, the rate of convergence of prices to equilibrium may be arbitrarily slow. The intuition for this is as follows: there may in general exist prices p at large distance from Walrasian equilibrium price p^* for which the excess demand is very small. This implies that the speed of adjustment of prices is very small, while the distance from equilibrium is very large.

However, under the assumption of strong monotonicity of demand, the Cauchy-Schwarz inequality implies that $\|p - p^*\| \leq \frac{1}{m}\|D(p) - s\|$, so prices cannot be large when excess demand is small. This implies that the price adjustment process cannot slow down at prices a long distance from Walrasian equilibrium.

The following result characterizes the rate of convergence of the continuous-time tâtonnement process to Walrasian equilibrium.

Proposition 7. *Consider the continuous-time tâtonnement process applied to a strongly monotone demand correspondence. The time to convergence to p within an ε-ball*[22] *of a Walrasian equilibrium price p^* is subpolynomial in ε.*

Proof. This involves a simple modification of the classical proof of convergence of tâtonnement for quasilinear economies. Consider the Lyapunov function for the differential equation

$$L(t) = \|p(t) - p^*\|^2.$$

By definition of the tâtonnement process,

$$\frac{dL}{dt} = 2(p(t) - p^*) \cdot \frac{dp}{dt} = 2(p(t) - p^*) \cdot \alpha(D(p(t)) - s).$$

By the definition of strong monotonicity,

$$\frac{dL}{dt} \geq -2\alpha m \|p(t) - p^*\|^2.$$

Solving this differential inequality gives

$$\|p(t) - p^*\| \leq e^{-2\alpha m t}.$$

But then for $t \geq \frac{-1}{2\alpha m} \log(\varepsilon)$, $\|p(t) - p^*\| \leq \varepsilon$.

□

This proof can also be adapted to the discrete-time version of the tâtonnement process.

Despite this, Budish, Cramton, Kyle, Lee, and Malec (2020) find that even under the strong monotonicity assumption, the tâtonnement algorithm may be too slow for practical identification of prices (in their setting, they hope to solve for prices in very large markets once per second). This illustrates the importance of the constant on the practical usefulness of the algorithm. Instead,

[22] For the analysis of tâtonnement as an algorithm, the ε-neighborhood of p^* is the appropriate subject of study. One reason as to why: p^* may be irrational, and thus one can never expect a computer to converge exactly to p^*.

Budish et al. (2020) find greater success in the use of an interior-point method for the convex program.

E Proofs omitted from the main text

E.1 Proof of Theorem 2

I begin with a helpful lemma.

Lemma 1. *Suppose $d \in D(p)$ and $d' \in D(p')$, with $(d - d') \cdot (p' - p) = 0$. Then $d \in D(p')$ and $d' \in D(p)$.*

Proof. Since $d \in D(p)$, by strong duality, from the dual objective associated with supply vector d,

$$\sum_{n \in N} u_n(p) + p \cdot d \leq \sum_{n \in N} u_n(p') + p' \cdot d.$$

Similarly,

$$\sum_{n \in N} u_n(p') + p' \cdot d' \leq \sum_{n \in N} u_n(p) + p \cdot d'.$$

Rearranging, and combining these inequalities,

$$(p' - p) \cdot d' \leq \sum_{n \in N} u_n(p) - \sum_{n \in N} u_n(p') \leq (p' - p) \cdot d.$$

However, by assumption $(p' - p) \cdot d' = (p' - p) \cdot d$, so that

$$\sum_{n \in N} u_n(p) - \sum_{n \in N} u_n(p') = (p' - p) \cdot d = (p' - p) \cdot d'.$$

Thus,

$$\sum_{n \in N} u_n(p) + p \cdot d = \sum_{n \in N} u_n(p') + p' \cdot d, \text{ and}$$
$$\sum_{n \in N} u_n(p) + p \cdot d' = \sum_{n \in N} u_n(p') + p' \cdot d'.$$

Strong duality then implies that $d \in D(p')$ and $d' \in D(p)$. □

I now proceed to the proof of the Theorem 2.

The sufficiency proof of Theorem 2 follows almost identically to the proof of Theorem 1, except that the demand selections on the left-hand side of (1) are replaced by selections from the total demand correspondence of the base economy, and the number of active buyers N_t^a on the right-hand side of (2) is replaced by the number of replicas k. Since k is $\Theta(|N_k|)$, the conclusion follows.

For necessity, consider the contrapositive: let $\mathscr{E}$ be a base economy which fails to be strongly monotone and let $D = \sum_{n \in N} D_n(p)$ be its total demand correspondence. Consider any real-valued sequence m_t with $m_t \to 0$, and let sequences of prices p_t, p'_t be such that $p_t \neq p'_t$ and $(d_t - d'_t) \cdot (p'_t - p_t) < m_t \|p_t - p'_t\|^2$ for $d_t \in D(p_t)$ and $d'_t \in D(p'_t)$ (the existence of such a sequence is assured by the failure of strong monotonicity). By the Bolzano-Weierstrass theorem, it is without loss to assume that $p_t \to p$ and $p'_t \to p'$ for some $p, p' \in \mathscr{P}$. There are two cases:

(1) $p \neq p'$. By Berge's Theorem, D is upper-hemicontinuous so that $d_t \to d \in D(p)$ and $d'_t \to d' \in D(p')$, and thus $(d - d') \cdot (p' - p) = 0$. Let $s = d$, then p must be a Walrasian equilibrium price in the sequence of economies $\mathscr{E}_k = \langle N_k, kd \rangle$. By Lemma 1, p' is a Walrasian equilibrium price for $\mathscr{E}_k$.

Without loss of generality,[23] consider any perturbation δs such that $\delta s \cdot (p' - p) > 0$. Note that p' cannot be an equilibrium price of the perturbed economies $\mathscr{E}'_k$ since

$$\sum_{n \in N_k} u_n(p) + p \cdot (kd + \delta s) - \left(\sum_{n \in N_k} u_n(p') + p' \cdot (kd + \delta s) \right) = \delta s \cdot (p - p') < 0.$$

On the other hand, in the limit as $k \to \infty$, the set of equilibrium prices of $\mathscr{E}'_k$ must approach a (closed, proper) subset of the equilibrium prices of the base economy (also the equilibrium prices of $\mathscr{E}_k$) since

$$\arg\min_p \sum_{n \in N_k} u_n(p) + p \cdot (kd + \delta s) = \arg\min_p \sum_{n \in N} u_n(p) + p \cdot \left(d + \frac{\delta s}{k} \right),$$

and the objective $\sum_{n \in N} u_n(p) + p \cdot (d + \delta s / k)$ epi-converges (as $k \to \infty$) to the objective of the unperturbed base economy, so that Theorem 7.33 of Rockafellar and Wets (2009) applies. But then p' is a Walrasian equilibrium price of $\mathscr{E}_k$ but not $\mathscr{E}'_k$, and $d_H(P_k, P'_k) \not\to 0$, so cannot be $O(1/|N_k|)$.

(2) $p = p'$. It suffices to consider the case when for all t, $d_t \neq d'_t$, otherwise the argument in the

[23] Relabeling p, p' if necessary.

previous case works as well. So, without loss of generality (restricting to a subsequence if necessary), assume that $p_t \to p$ and $p'_t \to p$ in such a way that the angle between $p'_t - p_t$ and $d'_t - d_t$ converges to a constant. *For now, assume that* $\sum_{n \in N} u_n(p) + d \cdot p$ *is twice continuously differentiable at* p. In this case, by assumption,

$$\lim_{t \to \infty} \frac{(d_t - d'_t) \cdot (p'_t - p_t)}{\|p_t - p'_t\|^2} = 0 \tag{*}$$

and this limit is the (negative of the) second directional derivative of $\sum_{n \in N} u_n(p)$ at p in the limiting direction of $p'_t - p_t$. That is, the failure of strong convexity implies a zero second derivative of the objective in some direction at some point.

I now argue that the limiting angle between $d'_t - d_t$ and $p_t - p'_t$ cannot be $90°$ (that is, the demand change cannot approach orthogonality with the price change). To see this, without loss of generality (changing orthonormal coordinates if necessary) suppose that $p_t - p'_t$ approaches unit vector in the direction of the first coordinate (say p_x) and $d_t - d'_t$ approaches the unit vector in the direction of the second coordinate (say p_y). In this case, $\frac{\partial d_x}{\partial p_x}(p) = 0$ and $\frac{\partial d_y}{\partial p_x}(p) \neq 0$. But then, by symmetry of the Slutsky matrix (equivalently, recognizing that these are mixed partials in the same coordinates and by Schwarz's Theorem), $\frac{\partial d_x}{\partial p_y}(p) \neq 0$. But this would imply that the Hessian of the objective at p is not positive semidefinite, which contradicts the convexity of the objective.

Thus (restricting to a subsequence if necessary), $(d_t - d'_t) \cdot (p'_t - p_t) \geq c\|d_t - d'_t\|\|p'_t - p_t\|$ for some $c > 0$ and for all t. Then (restricting to a subsequence if necessary), it is possible to take $\|d_t - d'_t\| = O(1/N)$ and the only way that

$$\frac{(d_t - d'_t) \cdot (p'_t - p_t)}{\|p_t - p'_t\|^2} \geq \frac{c\|d_t - d'_t\|}{\|p_t - p'_t\|}$$

can tend to zero is if $\|p_t - p'_t\| \geq \Omega(1/N)$.

I now adapt the argument to the case that $\sum u_n(p) + d \cdot p$ is not twice continuously differentiable at p. In this case, consider the sequence of $1/N^2$-Moreau-Yosida regularized economies (see Appendix E.2 below), notating the corresponding quantities in the regularized economies by tildes. Because $\tilde{d}_t - \tilde{d}'_t = d_t - d'_t + 1/N^2(p_t - p'_t)$ by Proposition 8 below, the limit in (*) must hold for a sequence of $\tilde{d}_t \in \tilde{D}(p), \tilde{d}'_t \in \tilde{D}(p')$ in the regularized economy. But then since the regularized objective is $C^{1,1}$ (continuous with Lipschitz continuous gradient), the limit of $(d_t - d'_t)/\|p_t - p'_t\|$ must approach a symmetric matrix by Theorem 13.52 of Rockafellar and Wets (2009) (a generalization of second-derivative symmetry for $C^{1,1}$ functions). The remainder of the above proof then follows through for the regularized economy. Finally, as argued in Appendix E.2, the sequence of regularized economies is perturbation-proof if and only if the original sequence is perturbation-proof.

E.2 Proof of Theorem 3

Notation I omit the t index except where necessary for clarity. Write $V(p) = \sum_{n \in N} u_n(p)$ for the realized total indirect utility and $D(p) = \partial V(p)$ for the realized demand correspondence. Recall that P is the set of minimizers of (D) which has the objective $V(p) + s \cdot p$, while P' is the set of minimizers of the objective $V(p) + (s + \delta s) \cdot p$. I abuse notation to write inequalities like $\|D(p) - D(p')\| \geq \|\delta s\|$ as shorthand for $\|d - d'\| \geq \|\delta s\|$ for all $d \in D(p)$ and $d' \in D(p')$.

Proof approach Consider economy $\mathscr{E} = \langle N, s \rangle$ obtained by drawing $N := |N|$ buyers from distribution ν over $\mathscr{V}$ which satisfies the conditions of Theorem 3.

My approach will be to show that with high probability (henceforth, w.h.p.) [24] over draws of the economy $\mathscr{E}$, that for *all* price vectors p with $\text{dist}(p, P) > c/N^{1-\varepsilon}$ (for a constant c to be chosen later), $\|D(p) - s\| > \|\delta s\|$. That is, w.h.p., the demand at prices p outside a neighborhood of size $c/N^{1-\varepsilon}$ from P must differ (in magnitude) from the supply vector s by more than the size of the perturbation $\|\delta s\|$. This will imply on that measure of economies that any price in P' must be within distance $c/N^{1-\varepsilon}$ of P, so that w.h.p. $d_H(P, P')$ will be less than $\frac{c}{N^{1-\varepsilon}}$.

Before completing the proof, I offer some high-level intuition for the approach and divide the proof into a number of steps.

1. **Concentration**: For any fixed p, p' at a distance of $c/N^{1-\varepsilon}$, I show using the Bernstein Inequality that w.h.p. $M(p, p') := \min_{d \in D(p), d' \in D(p')} (d - d') \cdot (p' - p)$ is at least $\frac{mN}{2}\|p - p'\|^2$. That is, w.h.p., the definition of strong monotonicity with constant $m/2$ holds for fixed prices p, p'. This will imply via the Cauchy-Schwarz Inequality logic used in Theorem 1 that for large enough c, w.h.p. $\|D(p) - D(p')\| > k\|\delta s\|$ for $k > 1$.

2. **Extension to discretized sphere**: Fixing p, I then extend the result that $\|D(p) - D(p')\| > k\|\delta s\|$ to *all* prices p' at distance of at least $c/N^{1-\varepsilon}$. To do so, I first discretize the $c/N^{1-\varepsilon}$

[24] Throughout, I use the term 'high probability' to refer to a probability that tends to 1 as $N \to \infty$. Then, X_t is $O_p(f(t))$ if $\left|\frac{X(t)}{f(t)}\right| < c$ w.h.p..

unit sphere and employ a union bound, which critically relies on the subexponential tail bound obtained from the Bernstein Inequality in Step 1.[25]

3. **Extension to sphere via regularization**: I then extend the result to the full $c/N^{1-\varepsilon}-$sphere centered at p under the assumption that the *realized* correspondence is Lipschitz continuous. At the end of the proof (in the paragraph titled *Regularization*), I show that this additional assumption is without loss of generality because in economies with non-Lipschitz demand correspondences, it is possible to analyze a *regularized* version of the economy with Lipschitz demand for which $d_H(P, P')$ is approximately (up to $o(1/N^{1-\varepsilon})$) equal to the original economy.

4. **Extension to exterior of sphere**: Using convexity, I then show that this implies $\|D(p) - D(p')\| > k\|\delta s\|$ for all p' with distance at least $c/N^{1-\varepsilon}$ from p.

5. **Uniformization over** p: Finally, I extend the result of Step 4 to a fine grid of prices over $\mathscr{P}$, and use the Lipschitzian property of demand in the regularized economy to establish that w.h.p. $\|D(p) - D(p')\| > k\|\delta s\|$ for all p, p' at distance of at least $c/N^{1-\varepsilon}$. This concludes the proof.

This proof resembles the "Fano method" used for proofs in the study of stochastic processes, but I have not been able to adapt known results to obtain these conclusions. I now fill in the details in these steps to complete the proof.

Step 1: Concentration Consider any fixed $p, p' \in \mathscr{P}$ with $\|p - p'\| \geq \frac{c}{N^{1-\varepsilon}}$. Define for each $n \in N$, $M_n(p, p') = \min_{d \in D_n(p), d' \in D_n(p')} (d - d') \cdot (p' - p)$ and let $M(p, p') = \sum_{n \in N} M_n(p, p')$. By the strong monotonicity assumption, $M_n(p, p')$ is a real-valued random variable satisfying $\mathbb{E}_\nu \left[M_n(p, p')\right] \geq m\|p - p'\|^2$. It will help to write $\mu_{p,p'} := \mathbb{E}_\nu \left[M_n(p, p')\right]$.

I apply the Bernstein Inequality[26]: given independent real-valued random variables $X_1, X_2, ..., X_N$ with $|X_i| \leq B$,

$$\Pr\left[\left|\sum_i X_i - \sum_i \mathbb{E}[X_i]\right| \geq t\right] \leq 2\exp\left(\frac{-\frac{1}{2}t^2}{\sum_i \mathbb{E}[X_i^2] + \frac{1}{3}Bt}\right).$$

To apply the Bernstein Inequality to $M_n(p, p')$, I require an estimate of the second moment of $M_n(p, p')$. I use the Bhatia and Davis (2000) inequality to obtain an upper bound: for any

[25] This is why the Bernstein Inequality is used rather than the simpler Chebyshev's Inequality, which is sufficient to obtain result in Step 1.

[26] See, for example Boucheron, Lugosi, and Massart (2013).

real-valued random variable X with mean μ and $m \le X \le M$ a.s., $\mathrm{Var}[X] \le (M-\mu)(\mu-m)$. Since $M_n(p,p')$ is bounded below by zero (by the monotonicity of d_n) and $M_n(p,p')$ is bounded a.s. above by $2X_{max}\|p-p'\|$ (using the Cauchy-Schwarz inequality), I have that

$$\mathbb{E}_\nu[M_n(p,p')^2] \le 2X_{max}\|p-p'\|\mu_{p,p'}.$$

Thus, applying the Bernstein Inequality to $M_n(p,p')$,

$$\begin{aligned}\Pr\left[M(p,p') \ge \frac{1}{2}N\mu_{p,p'}\right] &\ge 1-2\exp\left(\frac{-\frac{1}{8}N^2\mu_{p,p'}^2}{2NX_{max}\|p-p'\|\mu_{p,p'} + \frac{1}{3}NX_{max}\|p-p'\|\mu_{p,p'}}\right)\\ &= 1-2\exp\left(\frac{-3N\mu_{p,p'}}{56X_{max}\|p-p'\|}\right).\end{aligned}$$

Since $\mu_{p,p'} \ge m\|p-p'\|^2$ and $\|p-p'\| \ge c/N^{1-\varepsilon}$,

$$\begin{aligned}\Pr\left[M(p,p') \ge \frac{1}{2}mN\|p-p'\|^2\right] &\ge 1-2\exp\left(\frac{-3Nm\|p-p'\|^2}{56X_{max}\|p-p'\|}\right)\\ &\ge 1-2\exp\left(\frac{-3cN^\varepsilon m}{56X_{max}}\right)\end{aligned}$$

The above probability tends to 1 as $N \to \infty$. Note that the event $M(p,p') \ge \frac{mN}{2}\|p-p'\|^2$ for $\|p-p'\| = \frac{2k\|\delta s\|}{mN^{1-\varepsilon}}$ (that is, $c = 2k\|\delta s\|/m$, in my previous notation) is equivalent to the event that $(D(p)-D(p'))\cdot(p-p') \ge k\|\delta s\|N^\varepsilon\|p-p'\|$. By the Cauchy-Schwarz Inequality, this implies $\|d-d'\| \ge k\|\delta s\|N^\varepsilon$. For $k>1$ and sufficiently large N, if $p \in P$, this implies the event that p' could not be in P'. In later arguments, it will help to choose k larger than 1 to leave room for other approximations.

Step 2: Extension to discretized sphere With the same fixed p as in Step 1, I now consider $\mathbb{S}_c(p)$, the $c/N^{1-\varepsilon}$-sphere around p. By standard covering arguments, it is possible to identify $O(N^{(3+\varepsilon)L})$ points on the sphere of radius $O(1/N^{1-\varepsilon})$ such that the distance between each pair is at most $O(N^{-4})$. Let such a discretization be $\mathbb{D}_c(p)$.

Note that the number of pairs p, p' with $p' \in \mathbb{D}_c(p)$ is polynomial in N. A union bound over the events $M(p,p') \ge \frac{1}{2}mN\|p-p'\|^2$ over $p' \in \mathbb{D}_c(p)$ thus implies

$$\Pr\left[M(p,p') \ge \frac{1}{2}mN\|p-p'\|^2 \textbf{ for all } p' \in \mathbb{D}_c(p)\right] \ge 1-2O(N^{(3+\varepsilon)L})\exp\left(\frac{-3cN^\varepsilon m}{56X_{max}}\right),$$

which also tends to 1 as $N \to \infty$. Thus $\|D(p) - D(p')\| > k\|\delta s\|$ for all p' in $\mathbb{D}_c(p)$ w.h.p. for large enough N (where again, I have set $c = 2k\|\delta s\|/m$ in the above).

Assumption: In Steps 3 and 5, I assume that the realized demand correspondence is $O(N^2)$-Lipschitzian. I justify this assumption in my discussion on regularization below.

Step 3: Extension to sphere via regularization Consider $p'' \in \mathbb{S}_c(p) \setminus \mathbb{D}_c(p)$. Since p'' is at a distance of at most $O(N^{-4})$ from p' in $\mathbb{D}_c(p)$ and $(D(p) - D(p')) \cdot (p - p') \geq k\|\delta s\| N^\varepsilon \|p - p'\|$ w.h.p. for all p' in $\mathbb{D}_c(p)$, using the Cauchy Schwarz inequality,

$$
\begin{aligned}
&(D(p) - D(p'')) \cdot (p - p'') \\
&\geq (D(p) - D(p')) \cdot (p - p') - \|D(p') - D(p'')\|\|p - p'\| - \|D(p) - D(p')\|\|p' - p''\| \\
&\geq k\|\delta s\| N^\varepsilon \|p - p'\| - O(N^2) \cdot O(N^{-4})\|p - p'\| - \|D(p) - D(p')\| O(N^{-4}) \\
&\geq k'\|\delta s\| N^\varepsilon \|p - p'\|
\end{aligned}
$$

for any $k' < k$ and sufficiently large N, where the second line uses the $O(N^2)$-Lipschitz property of demand. This implies that all $p'' \in \mathbb{S}_c(p)$ have $\|D(p'') - D(p)\| \geq k'\|\delta s\|$ w.h.p. for sufficiently large N.

Step 4: Extension to exterior of sphere Now let p'' be a point outside of $\mathbb{S}_c(p)$ and let p' be the point on $\mathbb{S}_c(p)$ which is on the line between p and p''. By convexity, for all $d' \in D(p')$ and $d'' \in D(p'')$ that $(d'' - d') \cdot (p' - p'') \geq 0$. Since $p' - p'' = \frac{\|p' - p''\|}{\|p - p'\|}(p - p')$, $(d'' - d') \cdot (p - p') \geq 0$. But then since for all $d' \in D(p')$, $(d' - d) \cdot (p - p') \geq k'\|\delta s\| N^\varepsilon \|p - p'\|$ w.h.p., by adding the previous expression, $(d'' - d) \cdot (p - p') \geq k'\|\delta s\| N^\varepsilon \|p - p'\|$ with the same probability. But this implies that $\|d'' - d\| \geq k'\|\delta s\| N^\varepsilon$, as required. Thus, for all p' with $\|p - p'\| \geq \frac{c}{N^{1-\varepsilon}}$, $\|d - d'\| \geq k'\|\delta s\| N^\varepsilon$ with high probability.

Step 5: Uniformization over p Up until now, I have held p fixed, but I now extend the conclusion of Step 4 above to any realized $p \in P$. To do so, I apply another discretization of $\mathscr{P}$ with points at distance $\Theta(N^{-4})$. Again, by standard covering arguments (since $\mathscr{P}$ is compact), $O(N^{4L})$ points are required for such a covering of $\mathscr{P}$. A union bound again obtains the conclusions of Step 4 for *all* p in the discretization. Because the realized demand is $O(N^2)$-Lipschitz, this implies (via the same logic as in Step 3) the same result for $p \in P$ not in the covering for sufficiently large N.

This implies that with probability approaching 1 as $N \to \infty$, for any $p \in P$ and p' with $\|p - p'\| \geq \frac{c}{N^{1-\varepsilon}}$, that $\|s - d'\| \geq k'\|\delta s\| N^{\varepsilon}$ for all $d' \in D(p')$. By choosing c large enough for $k' > 1$, $\|s - d'\| > \delta s$ for all $d' \in D(p')$, which means p' cannot be in P'. This implies that $d_H(P, P') < \frac{c}{N^{1-\varepsilon}}$ with probability approaching 1 for sufficiently large c, that is the random variable $d_H(P, P')$ is $O_p\left(\frac{1}{N^{1-\varepsilon}}\right)$, as is required.

Regularization In Steps 3 and 5 above, I assumed that the realized demand correspondence is an $O(N^2)-$Lipschitz. Here, I show that this assumption is without loss of generality by exploiting the Moreau-Yosida regularization of convex functions. The explicit construction of the Moreau-Yosida approximation will not be important for my argument (although it is not complicated—see, for example, Rockafellar and Wets (2009)), so instead I state the result as an existence theorem.

Proposition 8 (Moreau-Yosida)**.** *Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper, convex and lower semi-continuous function defined on a convex, compact subset X of a Hilbert space. Then for all $\gamma > 0$, there exists a function $\tilde{f} : X \to \mathbb{R} \cup \{\infty\}$, the* γ-Moreau envelope *of f, with the following properties:*

- *$\tilde{f}$ is convex, $\frac{1}{\gamma}$-Lipschitz-continuous and Fréchet-differentiable with gradient $\nabla \tilde{f}$ which is $\frac{1}{\gamma}$-Lipschitz continuous*
- *f and $\tilde{f}$ have the same minimizers.*

Furthermore, if f is L-Lipschitz continuous, then $\tilde{f}$ is also L-Lipschitz, and for all $x \in X$,

$$\tilde{f}(x) \leq f(x) \leq \tilde{f}(x) + \frac{\gamma L^2}{2}.$$

The inverse mapping of the gradient of $\tilde{f}$ and the inverse mapping of the subdifferential of f are related by

$$(\nabla \tilde{f})^{-1}(x^*) = \gamma x^* + (\partial f)^{-1}(x^*).$$

Note that V is proper, convex and $X_{max}-$Lipschitz, where X_{max} is defined as the maximum magnitude demand vector, $\max_{v_n \in \text{supp}(\nu)} \max_{x \in \text{dom}(v_n)} \|x\|$ (which exists by the assumption of compactness of the consumption possibility sets). The $\frac{1}{N^2}-$Moreau envelope of V, $\tilde{V}$, is thus convex, $\max\{X_{max}, N^2\}$-Lipschitz continuous and Fréchet differentiable with gradient (i.e. demand function) which is N^2-Lipschitz.

I now show that it suffices to analyze the $1/N^2$ regularized dual objective. Let $\tilde{P}$ and $\tilde{P}'$ be the unperturbed and perturbed Walrasian prices (respectively) for the regularized demand.

First, note that $d_H(P,P') = d_H(\tilde{P},\tilde{P}') + O(1/N^2)$, which implies that if $d_H(\tilde{P},\tilde{P}')$ is $O(1/N^{1-\varepsilon})$, so is $d_H(P,P')$. To see this, note that $\tilde{P} = (\nabla\tilde{V})^{-1}(s)$, $\tilde{P}' = (\nabla\tilde{V})^{-1}(s+\delta s)$, $P = (\partial V)^{-1}(s)$ and $P' = (\partial V)^{-1}(s+\delta s)$ so that by the last identity in Proposition 8, $\tilde{P} = P + \frac{1}{N^2}s$ and $\tilde{P}' = P' + \frac{1}{N^2}(s+\delta s)$ which, since $\|s\| < X_{max}$ and $\|s+\delta s\| < X_{max}$, implies the first claim.

Second, I claim that $\mathbb{E}[\nabla\tilde{V}]$ is m'-strongly monotone for all $m' < m$ and sufficiently large N and $\|p - p'\| \geq \frac{c}{N^{1-\varepsilon}}$. To see this, consider the expression $\mathbb{E}[(p-p')\cdot(\tilde{d}'-\tilde{d})]$ for $\tilde{d} = \nabla V(p)$ and $\tilde{d}' = \nabla V(p')$. Note $p \in (\nabla\tilde{V})^{-1}(\tilde{d})$ and $p' \in (\nabla\tilde{V})^{-1}(\tilde{d}')$ so that $p - \frac{1}{N^2}\tilde{d} \in (\partial V)^{-1}(\tilde{d})$ and $p' - \frac{1}{N^2}\tilde{d}' \in (\partial V)^{-1}(\tilde{d}')$. But then

$$\begin{aligned}
\mathbb{E}[(p-p')\cdot(\tilde{d}'-\tilde{d})] &= \mathbb{E}[(p - \frac{1}{N^2}\tilde{d} - p' + \frac{1}{N^2}\tilde{d}')\cdot(\tilde{d}'-\tilde{d}) + \frac{1}{N^2}(\tilde{d}-\tilde{d}')\cdot(\tilde{d}'-\tilde{d})] \\
&\geq m\|p - \frac{1}{N^2}\tilde{d} - p' + \frac{1}{N^2}\tilde{d}'\|^2 - \frac{1}{N^2}\|\tilde{d}-\tilde{d}'\|^2 \\
&\geq m\|p-p'\|^2 - \frac{m}{N^2}\|\tilde{d}-\tilde{d}'\|^2 - \frac{1}{N^2}\|\tilde{d}-\tilde{d}'\|^2 \\
&\geq m\|p-p'\|^2 - \frac{1+m}{N^2}X_{max}^2,
\end{aligned}$$

where the second line above follow by the strong monotonicity property of ∂V. So, for $\|p-p'\| \geq \frac{c}{N^{1-\varepsilon}}$ the second term is asymptotically dominated by the first, and the claim follows. Together, these two claims imply that I may replicate the arguments in Steps 1-2 above for the regularized demand $\nabla\tilde{V}$ (which is Lipschitz) and Steps 3-5 imply the required result.

E.3 Proof of Theorem 4

Part (a) follows by direct combination of Theorem 3 with Proposition 2.

For part (b), note that in the proof of Theorem 3, I showed that with subexponential probability—in fact, with probability $1 - O(1/N)$, the maximum distance between the price associated with the truthful report of an agent and any alternative report of that agent is $O(1/N^{1-\varepsilon})$. In the complementary $O(1/N)$ measure of draws of economies, the maximum influence on price is $O(1)$, since by assumption the set of possible prices $\mathscr{P}$ is compact. Thus, the expected maximum influence of any report, including the *interim* optimal report, on price is $(1-O(1/N))O(1/N^{1-\varepsilon}) + O(1/N)O(1) = O(1/N^{1-\varepsilon})$

E.4 Proof of Proposition 3

I use some concepts introduced by Baldwin and Klemperer (2019) and refer readers to Baldwin and Klemperer (2019) for a complete treatment.

Definition E.1. For buyer n with demand correspondence D_n, the *locus of indifference prices (LIP)* is $\mathscr{L}_n = \left\{ p \in \mathbb{R}^L_+ : |D_n(p)| > 1 \right\}$.

The LIP divides price space into *unique demand regions* in which demand is constant, so that demand can only change as prices change through the $(L-1)-$dimensional facets that comprise the LIP.[27] Moreover, Baldwin and Klemperer (2019) show that as prices change between adjacent unique demand regions, demand changes by an integer multiple of the "primitive" normal vector of the associated facet(s) separating the regions. Here, a primitive vector is one in which the greatest common divisor of its entries is 1.

Definition E.2. The *demand type* $\mathscr{D}_n$ of buyer n is the set of primitive facet normal vectors of the LIP. I refer to an element of $\mathscr{D}_n$ as a *demand subtype*.[28]

Consider any price change $p \mapsto p'$. For all the demand subtypes δ associated with buyers in $\mathscr{V}$ (note there are finitely many possible subtypes for L goods with bounded demand), either $\delta \cdot (p' - p) = 0$ or $\delta \cdot (p' - p) > 0$. In the first case, p and p' must both lie on the same facet of the LIP, so that demand does not change along p to p'. In the other case, since the number of possible demand subtypes is finite, there is a least $\delta \cdot (p' - p)$ among them: let δ' be that subtype and let $\delta' \cdot (p' - p) = k\|p' - p\|$ for some $k > 0$. (In other words, k is the least product of $\|\delta\|$ among the demand subtypes and the cosine of the angle between δ and $p' - p$. This is why my expression maintains $\|p' - p\|$ as a constant of proportionality.)

For now, suppose that $\alpha\|p - p'\| \leq 1$. In this case, a lower bound on $\mathbb{E}[(D(p) - D(p')) \cdot (p' - p)]$ is given by the $\Pr_\nu[D(p) \neq D(p')]$ multiplied by the least value of $(D(p) - D(p')) \cdot (p' - p)$ conditional on a demand change. Since by assumption $\Pr_\nu[D(p) \neq D(p')] \geq \alpha\|p - p'\|$ and the least value of the projected demand change is $k\|p' - p\|$, a lower bound on $\mathbb{E}[(D(p) - D(p')) \cdot (p' - p)]$ is $\alpha k\|p' - p\|^2$, which is the required inequality for strong monotonicity.

If $\alpha\|p - p'\| \geq 1$, divide the line segment up into pieces $p, p_1, p_2, ..., p_N, p'$ where between p and p_1, p_1 and p_2, p_2 and p_3 etc., demand changes occur with probability 1, and between p_N and p', demand

[27] Note that the cyclic monotonicity of demand implies that the change in demand as p changes to p' is independent of the path in price space between p and p'. So, unless otherwise specified, a price change from p to p' means a straight line path between p and p'.

[28] Note that the "subtype" terminology is not used by Baldwin and Klemperer (2019).

changes occur with at least $\alpha\|p_N - p'\|$. In this case, since the demand changes are lower bounded by the size of an indivisibility, it is still clear that the size of the demand change is proportional to the distance $\|p - p'\|$, as is required for strong monotonicity in this setting.