Who Gets What and When: Dynamic Allocation without Transfers

Mitchell Watt

October 21, 2024

Introduction

In many important markets, items are not allocated at once but instead arrive over time.

Introduction

In many important markets, items are not allocated at once but instead arrive over time.

- Public housing units (Arnosti and Shi 2019, Cook and Li (202x))
- Organs (Roth et al. (2004), Akbarpour et al. (2020), Ashlagi et al. (2021)
- Uber drivers waiting for jobs (Castro et al. 2021)
- Centralized allocation of schoolteachers (Combe, Tercieux and Terrier 2022)

Introduction

In many important markets, items are not allocated at once but instead arrive over time.

- Public housing units (Arnosti and Shi 2019, Cook and Li (202x))
- Organs (Roth et al. (2004), Akbarpour et al. (2020), Ashlagi et al. (2021)
- Uber drivers waiting for jobs (Castro et al. 2021)
- Centralized allocation of schoolteachers (Combe, Tercieux and Terrier 2022)

Common features: queues, costs of waiting, fixed or constrained prices

Previous work: optimal one-off queue design, value of thickness

This project: focused on repeated allocation, optimal dynamic contracting

Motivating examples

Ride-sharing apps

- Jobs appear randomly over time.
- Net of payments, Castro et al. (2021) show substantial heterogeneity in values of jobs.
- Typically allocated FCFS.

Motivating examples

Allocation of school teachers

- Teachers allocated centrally.
- Jobs appear over time: retirement of teachers, new demand.
- Heterogeneity in value: locational preferences and difficult schools
- Regulatory limits on salary differential.
- "Transfer points": teachers accrue priority while matched to less desirable schools.

▸ More details

This paper

Key question

What is the optimal design of dynamic incentives in matching markets with fixed transfers?

This paper

Key question

What is the optimal design of dynamic incentives in matching markets with fixed transfers?

I introduce a model of repeated matching with a fixed population of agents and a period-by-period participation constraint.

Key assumptions: values are homogeneous and observable, non-stochastic agent arrival

This paper

Key question

What is the optimal design of dynamic incentives in matching markets with fixed transfers?

I introduce a model of repeated matching with a fixed population of agents and a period-by-period participation constraint.

Key assumptions: values are homogeneous and observable, non-stochastic agent arrival

Key results:

- Principal incentivizes undesirable allocations using promises of improved future allocations.
- Principal's value function is Schur-concave in promised utility vector.
  - $\rightarrow$ Loyalty: agents with worse historical allocations prioritized for better allocations today.

Roadmap

Single agent model

Single agent optimal contract

Multiple agents

Conclusion and next steps

Model

Agents and timing

One agent and a principal.

Time is discrete, $t \in \mathbb{N}$.

Model

Agents and timing

One agent and a principal.

Time is discrete, $t \in \mathbb{N}$. In each period:

- An indivisible item arrives, with value is drawn i.i.d. from common knowledge F with support $\mathcal{V} = [\underline{v}, \overline{v}]$ containing zero.

Model

Agents and timing

One agent and a principal.

Time is discrete, $t \in \mathbb{N}$. In each period:

- An indivisible item arrives, with value is drawn i.i.d. from common knowledge F with support $\mathcal{V} = [\underline{v}, \overline{v}]$ containing zero.
- The principal and agent both observe the item's value.

Model

Agents and timing

One agent and a principal.

Time is discrete, $t \in \mathbb{N}$. In each period:

- An indivisible item arrives, with value is drawn i.i.d. from common knowledge F with support $\mathcal{V} = [\underline{v}, \overline{v}]$ containing zero.
- The principal and agent both observe the item's value.
- Principal offers the item to the agent with probability $x(v) \in [0, 1]$.

Model

Agents and timing

One agent and a principal.

Time is discrete, $t \in \mathbb{N}$. In each period:

- An indivisible item arrives, with value is drawn i.i.d. from common knowledge F with support $\mathcal{V} = [\underline{v}, \overline{v}]$ containing zero.
- The principal and agent both observe the item's value.
- Principal offers the item to the agent with probability $x(v) \in [0, 1]$.
- Agent accepts the offered item with probability $y(v) \in [0, 1]$.

Model

Agents and timing

One agent and a principal.

Time is discrete, $t \in \mathbb{N}$. In each period:

- An indivisible item arrives, with value is drawn i.i.d. from common knowledge F with support $\mathcal{V} = [\underline{v}, \overline{v}]$ containing zero.
- The principal and agent both observe the item's value.
- Principal offers the item to the agent with probability $x(v) \in [0, 1]$.
- Agent accepts the offered item with probability $y(v) \in [0, 1]$.
- Unallocated / unaccepted items disappear.

Model

Preferences

v_t is the value of the item arriving in period t.

y_t is agent i's acceptance decision at period t

δ_A is agent's discount factor, δ_P is principal's discount factor.

Model

Preferences

v_t is the value of the item arriving in period t.

y_t is agent i's acceptance decision at period t

δ_A is agent's discount factor, δ_P is principal's discount factor.

Agent utility

$$U^A = (1 - \delta_A) \sum_{t=0}^{\infty} \delta_A^t v_t y_t.$$

Principal utility

$$U^P = (1 - \delta_P) \sum_{t=0}^{\infty} \delta_P^t y_t.$$

Design problem

Principal chooses a sequence of history-dependent allocation rules.

$$x_t : \mathcal{V} \times \mathcal{H}_t \to [0,1].$$

For now, assume the principal has full commitment.

Agent chooses probability of acceptance conditional on offer value

$$y_{it} : \mathcal{V} \times \mathcal{H}_t \to [0,1].$$

Design problem

Principal chooses a sequence of history-dependent allocation rules.

$$x_t : \mathcal{V} \times \mathcal{H}_t \to [0,1].$$

For now, assume the principal has full commitment.

Agent chooses probability of acceptance conditional on offer value

$$y_{it} : \mathcal{V} \times \mathcal{H}_t \to [0,1].$$

Lemma

There is an optimal mechanism with no randomization.

Design problem

Principal chooses a sequence of history-dependent allocation rules.

$$x_t : \mathcal{V} \times \mathcal{H}_t \to \{0, 1\}.$$

For now, assume the principal has full commitment.

Agent chooses probability of acceptance conditional on offer value

$$y_{it} : \mathcal{V} \times \mathcal{H}_t \to \{0, 1\}.$$

Design problem

Recursive formulation

Recall by the Bellman (1952) Principle of Optimality, it suffices for the policy to depend on history only through the *promised utility* to the agent, u.

Design problem

Recursive formulation

Recall by the Bellman (1952) Principle of Optimality, it suffices for the policy to depend on history only through the *promised utility* to the agent, u.

As a function of u and the realization of the item's value, the principal determines an allocation rule $x(v;u)$ and a plan for new promised utilities $u'(v;u)$.

Design problem

Recursive formulation

Recall by the Bellman (1952) Principle of Optimality, it suffices for the policy to depend on history only through the *promised utility* to the agent, u.

As a function of u and the realization of the item's value, the principal determines an allocation rule $x(v;u)$ and a plan for new promised utilities $u'(v;u)$.

Since promises must be realized by a stream of future allocations,

$$u \in \left[0, \int_0^{\overline{v}} v \, \mathrm{d}F(v)\right] := \mathcal{U}.$$

Design problem

Recursive reformulation via Bellman (1952)

The principal solves:

$$\Phi(u) = \max_{x(v;u),u'(v;u)} \mathbb{E}_{v\sim F}\left[(1-\delta_P)x(v;u) + \delta_P\Phi\left(u'(v;u)\right)\right] \text{ subject to}$$

Design problem

Recursive reformulation via Bellman (1952)

The principal solves:

$$\Phi(u) = \max_{x(v;u),u'(v;u)} \mathbb{E}_{v\sim F}\left[(1-\delta_P)x(v;u) + \delta_P \Phi\left(u'(v;u)\right)\right] \text{ subject to}$$

$$\mathbb{E}_{v\sim F}\left[(1-\delta_A)vx(v;u) + \delta_A u'(v;u)\right] \geq u, \qquad \text{(PK)}$$

Design problem

Recursive reformulation via Bellman (1952)

The principal solves:

$$\Phi(u) = \max_{x(v;u),u'(v;u)} \mathbb{E}_{v\sim F}\left[(1-\delta_P)x(v;u) + \delta_P \Phi\left(u'(v;u)\right)\right] \text{ subject to}$$

$$\mathbb{E}_{v\sim F}\left[(1-\delta_A)vx(v;u) + \delta_A u'(v;u)\right] \geq u, \qquad \text{(PK)}$$

$$(1-\delta_A)vx(v;u) + \delta_A u'(v;u) \geq 0, \text{ for each } v, \qquad \text{(PC)}$$

Design problem

Recursive reformulation via Bellman (1952)

The principal solves:

$$\Phi(u) = \max_{x(v;u),u'(v;u)} \mathbb{E}_{v\sim F}\left[(1-\delta_P)x(v;u) + \delta_P \Phi\left(u'(v;u)\right)\right] \text{ subject to}$$

$$\mathbb{E}_{v\sim F}\left[(1-\delta_A)vx(v;u) + \delta_A u'(v;u)\right] \geq u, \tag{PK}$$

$$(1-\delta_A)vx(v;u) + \delta_A u'(v;u) \geq 0, \text{ for each } v, \tag{PC}$$

$$\text{with } x(v;u) \in \{0,1\} \text{ and } u'(v;u) \in \mathcal{U}.$$

Roadmap

Single agent model

Single agent optimal contract

Multiple agents

Conclusion and next steps

Value function properties

Theorem

There is a unique value function $\Phi(\cdot)$ solving the principal's problem, which is monotone decreasing, concave, continuous and semidifferentiable.

Value function properties

Proof idea

$$\Phi(u) = \max_{x(v;u),u'(v;u)} \mathbb{E}_{v\sim F}\left[(1-\delta_P)x(v;u) + \delta_P \Phi\left(u'(v;u)\right)\right] \text{ subject to}$$

$$\mathbb{E}_{v\sim F}\left[(1-\delta_A)vx(v;u) + \delta_A u'(v;u)\right] \geq u, \tag{PK}$$

$$(1-\delta_A)vx(v;u) + \delta_A u'(v;u) \geq 0, \text{ for each } v, \tag{PC}$$

$$\text{with } x(v;u) \in \{0,1\} \text{ and } u'(v;u) \in \mathcal{U}.$$

- Blackwell's conditions: RHS operator is a contraction.

Value function properties

Proof idea

$$\Phi(u) = \max_{x(v;u), u'(v;u)} \mathbb{E}_{v\sim F}\left[(1-\delta_P)x(v;u) + \delta_P \Phi\left(u'(v;u)\right)\right] \text{ subject to}$$

$$\mathbb{E}_{v\sim F}\left[(1-\delta_A)vx(v;u) + \delta_A u'(v;u)\right] \geq u, \tag{PK}$$

$$(1-\delta_A)vx(v;u) + \delta_A u'(v;u) \geq 0, \text{ for each } v, \tag{PC}$$

$$\text{with } x(v;u) \in \{0,1\} \text{ and } u'(v;u) \in \mathcal{U}.$$

- Blackwell's conditions: RHS operator is a contraction.
- Endomorphism on space of concave functions:

Value function properties

Proof idea

$$\Phi(u) = \max_{x(v;u),u'(v;u)} \mathbb{E}_{v\sim F}\left[(1-\delta_P)x(v;u) + \delta_P \Phi\left(u'(v;u)\right)\right] \text{ subject to}$$

$$\mathbb{E}_{v\sim F}\left[(1-\delta_A)vx(v;u) + \delta_A u'(v;u)\right] \geq u, \tag{PK}$$

$$(1-\delta_A)vx(v;u) + \delta_A u'(v;u) \geq 0, \text{ for each } v, \tag{PC}$$

$$\text{with } x(v;u) \in \{0,1\} \text{ and } u'(v;u) \in \mathcal{U}.$$

- Blackwell's conditions: RHS operator is a contraction.
- Endomorphism on space of concave functions:
  - → For $u^\alpha = \alpha u + (1-\alpha)u^*$, feasible to assign using $x(v;u)$, $u'(v;u)$ w.p. α and $x(v;u^*)$, $u'(v;u^*)$ w.p. $1-\alpha \Rightarrow$ Jensen's inequality.

Value function properties

Proof idea

$$\Phi(u) = \max_{x(v;u),u'(v;u)} \mathbb{E}_{v\sim F}\left[(1-\delta_P)x(v;u) + \delta_P \Phi\left(u'(v;u)\right)\right] \text{ subject to}$$

$$\mathbb{E}_{v\sim F}\left[(1-\delta_A)vx(v;u) + \delta_A u'(v;u)\right] \geq u, \qquad \text{(PK)}$$

$$(1-\delta_A)vx(v;u) + \delta_A u'(v;u) \geq 0, \text{ for each } v, \qquad \text{(PC)}$$

$$\text{with } x(v;u) \in \{0,1\} \text{ and } u'(v;u) \in \mathcal{U}.$$

- Blackwell's conditions: RHS operator is a contraction.
- Endomorphism on space of concave functions:
  → Banach fixed point theorem ⇒ concavity.

Value function properties

Proof idea

$$\Phi(u) = \max_{x(v;u),u'(v;u)} \mathbb{E}_{v\sim F}\left[(1-\delta_P)x(v;u) + \delta_P \Phi\left(u'(v;u)\right)\right] \text{ subject to}$$

$$\mathbb{E}_{v\sim F}\left[(1-\delta_A)vx(v;u) + \delta_A u'(v;u)\right] \geq u, \tag{PK}$$

$$(1-\delta_A)vx(v;u) + \delta_A u'(v;u) \geq 0, \text{ for each } v, \tag{PC}$$

$$\text{with } x(v;u) \in \{0,1\} \text{ and } u'(v;u) \in \mathcal{U}.$$

- Monotonicity: set of feasible policies is decreasing in u.
- Continuity and semidifferentiability: interior continuity and semidifferentiability follow from concavity, continuity at end points from a limit argument. □

Characterizing the optimal allocation

Cutoff policy

Lemma

There is an optimal policy in which

$$x(v;u) = \begin{cases} 0 & \text{if } v < \gamma(u) \\ 1 & \text{if } v \geq \gamma(u), \end{cases}$$

for some $\gamma : \mathcal{U} \to \mathcal{V}$.

Intuition: If otherwise, the agent would prefer to be allocated the same proportion of goods but with higher values, and the principal is indifferent (and may reduce some u').

Characterizing the optimal allocation

Promises

Suppose we fix cutoff policy $\gamma(u)$. We now determine the optimal promise policy $u'(v; u)$.

Characterizing the optimal allocation

Promises

Suppose we fix cutoff policy $\gamma(u)$. We now determine the optimal promise policy $u'(v; u)$.

At each $v \geq \gamma(u)$, (PC) requires $u'(v; u) \geq \frac{-(1-\delta_A)}{\delta_A} v$.

Characterizing the optimal allocation

Promises

Suppose we fix cutoff policy $\gamma(u)$. We now determine the optimal promise policy $u'(v;u)$.

At each $v \geq \gamma(u)$, (PC) requires $u'(v;u) \geq \frac{-(1-\delta_A)}{\delta_A}v$.

Averaged over all $v \in \mathcal{V}$, (PK) requires

$$\mathbb{E}_{v\sim F}[u'(v;u)] \geq \frac{u-(1-\delta_A)\int_{\gamma(u)}^{\overline{v}} v dF(v)}{\delta_A}.$$

Characterizing the optimal allocation

Promises

Suppose we fix cutoff policy $\gamma(u)$. We now determine the optimal promise policy $u'(v;u)$.

At each $v \geq \gamma(u)$, (PC) requires $u'(v;u) \geq \frac{-(1-\delta_A)}{\delta_A}v$.

Averaged over all $v \in \mathcal{V}$, (PK) requires

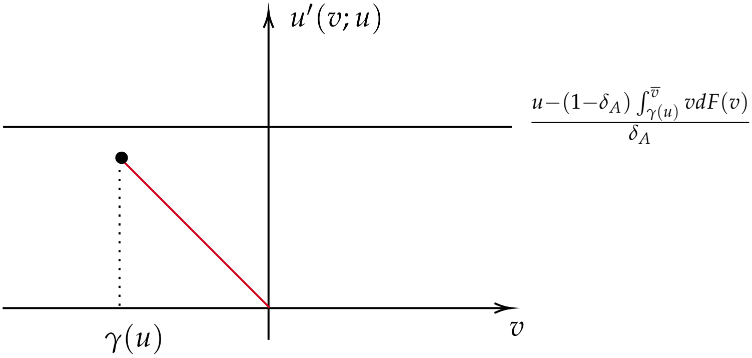
$$\mathbb{E}_{v\sim F}[u'(v;u)] \geq \frac{u - (1-\delta_A)\int_{\gamma(u)}^{\overline{v}} v dF(v)}{\delta_A}.$$

The concavity of $\Phi(\cdot)$ implies it is optimal to attain the average on the right in the least spread way, while respecting (PC).

Characterizing the optimal allocation

Promises

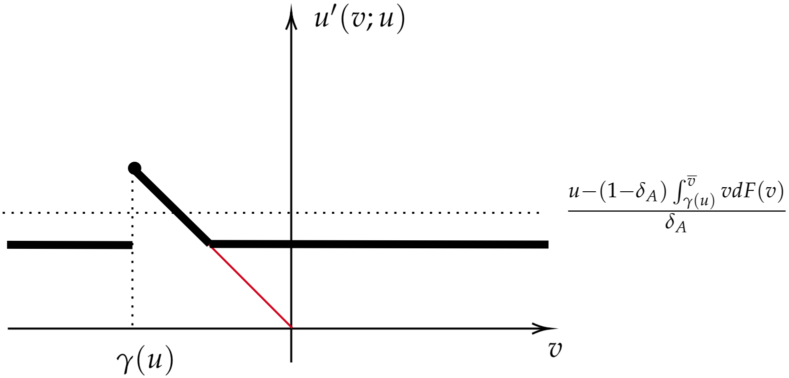
First possibility: no participation constraints bind, constant u'.



Characterizing the optimal allocation

Promises

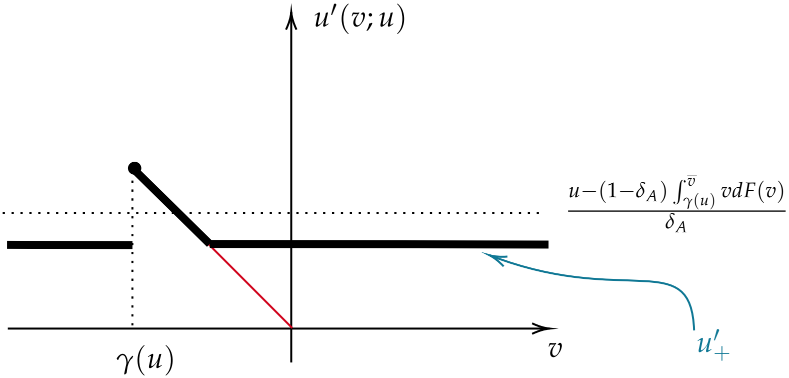
Second possibility: an interval of binding participation constraints and constant u' elsewhere.



Characterizing the optimal allocation

Promises

Second possibility: an interval of binding participation constraints and constant u' elsewhere.



Optimal allocations depend on δ_A vs δ_P

Principal chooses cutoffs trading off (using δ_P) the probability of allocating today and the effect on future promises (which depend on δ_A).

Optimal allocations depend on δ_A vs δ_P

Principal chooses cutoffs trading off (using δ_P) the probability of allocating today and the effect on future promises (which depend on δ_A).

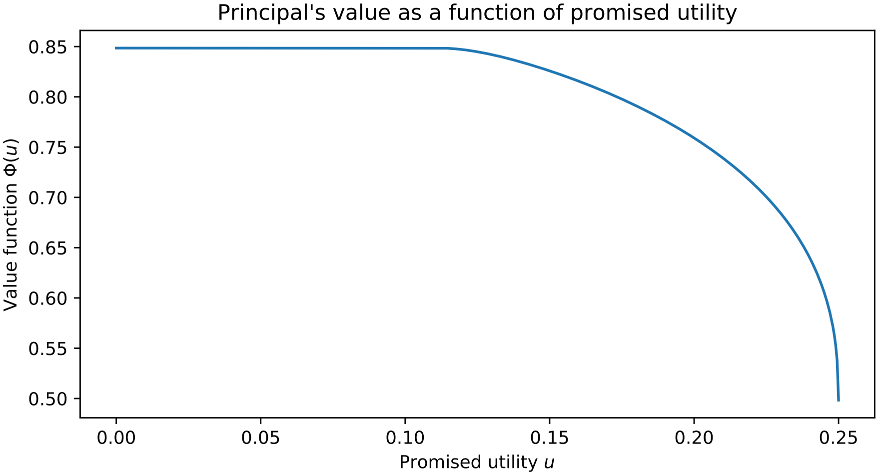
Theorem (Informal)

The optimal allocation entails a cutoff policy $\gamma(\cdot)$ nondecreasing with $\gamma(\max \mathcal{U}) = 0$. Whenever (PK) binds:

- *When $\delta^P > \delta^A$, $u'_+ < u$. The principal "works off" promises over time, and the cutoff (thus expected value of allocated items) fluctuates (inversely) with promises.*
- *When $\delta^P = \delta^A$, $u'_+ = u$. Eventually, the allocation rule is deterministic with cutoff < 0.*
- *When $\delta^P < \delta^A$, $u'_+ > u$. Eventually only good items are allocated.*

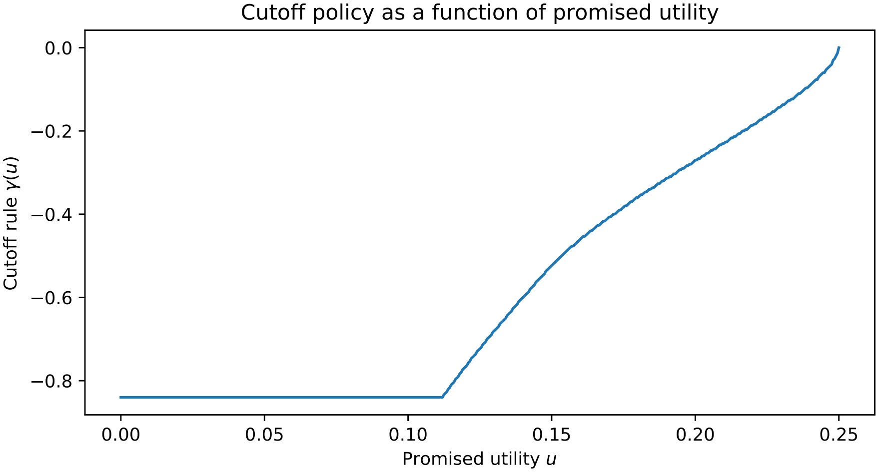
Example - Patient principal

$v \sim \mathrm{Unif}[-1, 1]$, $\mathcal{U} = \left[0, \frac{1}{4}\right]$, $\delta_P = 0.9 > 0.8 = \delta_A$



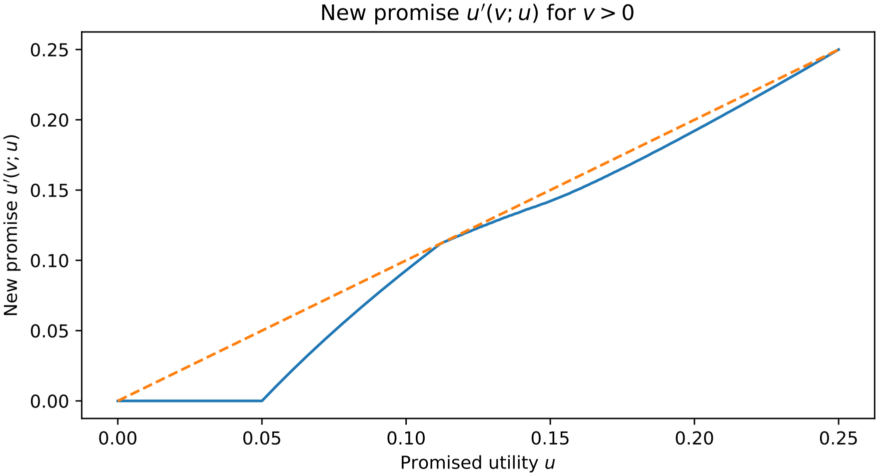
Example - Patient principal

$v \sim \text{Unif}[-1, 1]$, $\mathcal{U} = \left[0, \frac{1}{4}\right]$, $\delta_P = 0.9 > 0.8 = \delta_A$



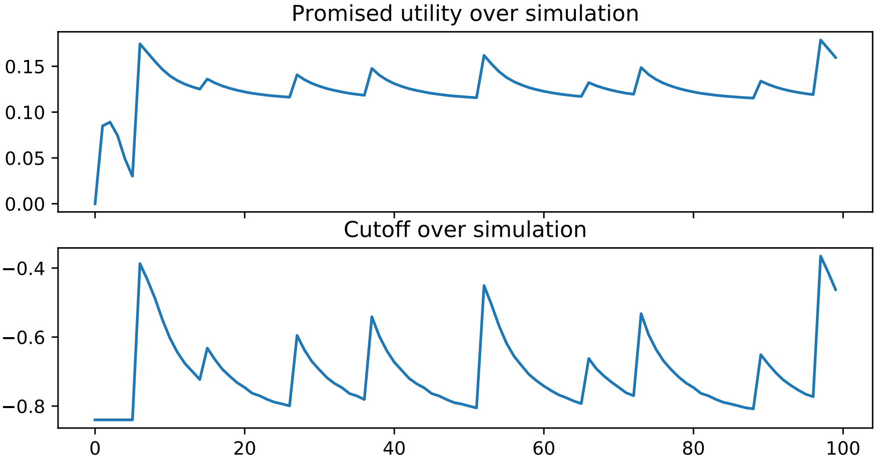
Example - Patient principal

$v \sim \text{Unif}[-1,1]$, $\mathcal{U} = \left[0, \frac{1}{4}\right]$, $\delta_P = 0.9 > 0.8 = \delta_A$



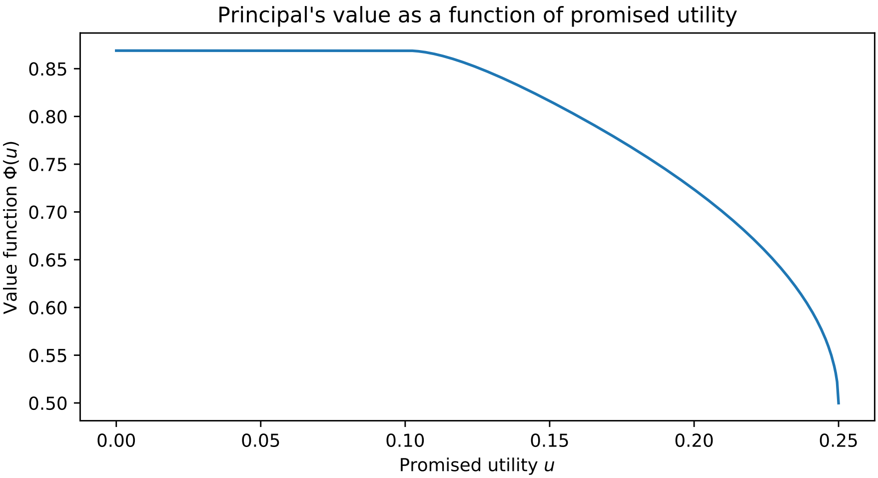
Example - Patient Principal - Dynamics

$v \sim \text{Unif}[-1,1]$, $\mathcal{U} = \left[0, \frac{1}{4}\right]$, $\delta_P = 0.9 > 0.8 = \delta_A$



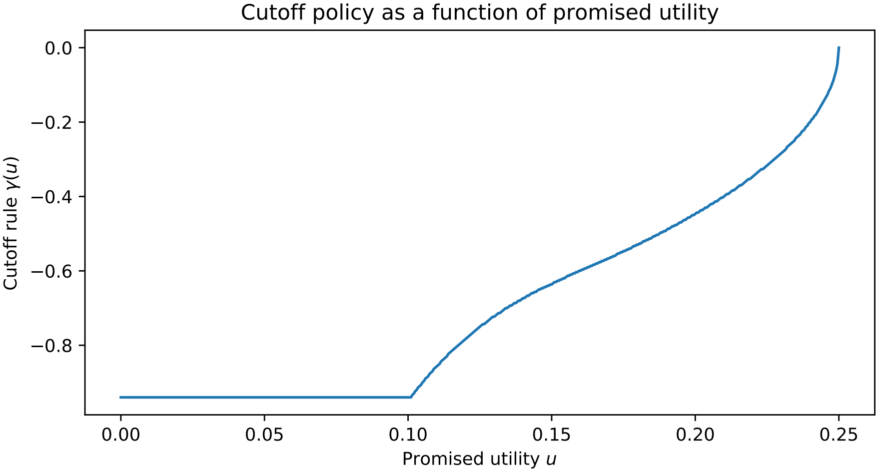
Example - Equally patient principal and agent

$v \sim \mathrm{Unif}[-1, 1]$, $\mathcal{U} = \left[0, \frac{1}{4}\right]$, $\delta_P = 0.8 = \delta_A$



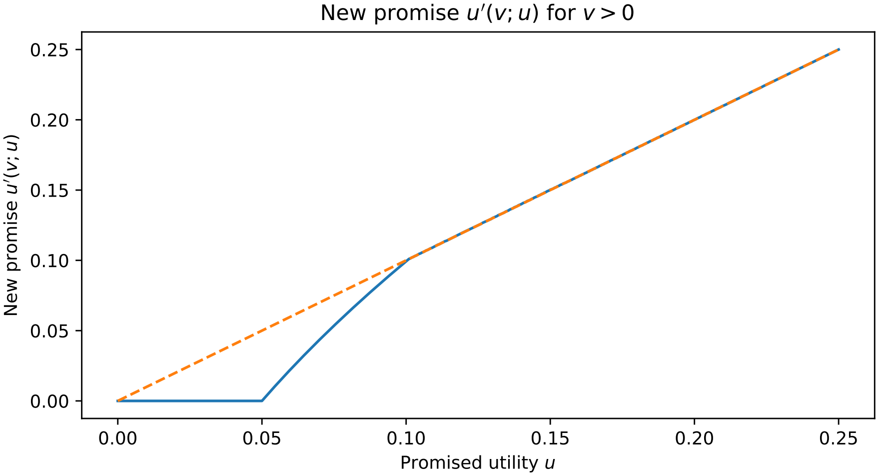
Example - Equally patient principal and agent

$v \sim \text{Unif}[-1, 1]$, $\mathcal{U} = \left[0, \frac{1}{4}\right]$, $\delta_P = 0.8 = \delta_A$



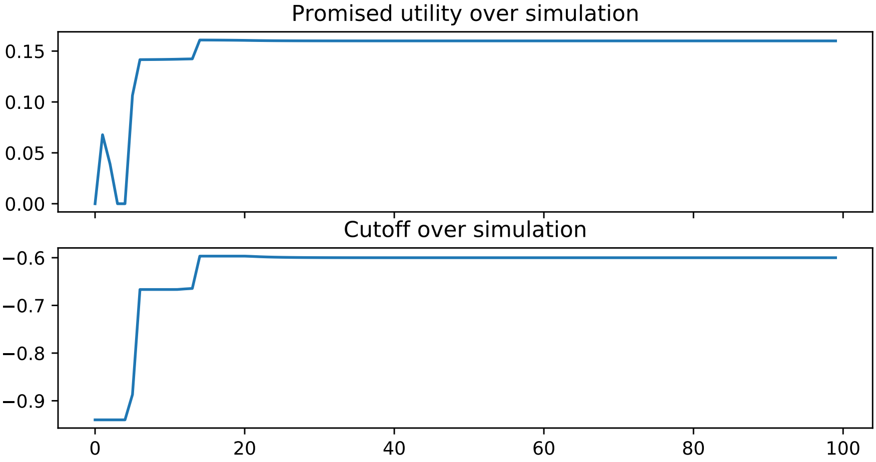
Example - Equally patient principal and agent

$v \sim \text{Unif}[-1,1]$, $\mathcal{U} = \left[0, \frac{1}{4}\right]$, $\delta_P = 0.8 = \delta_A$



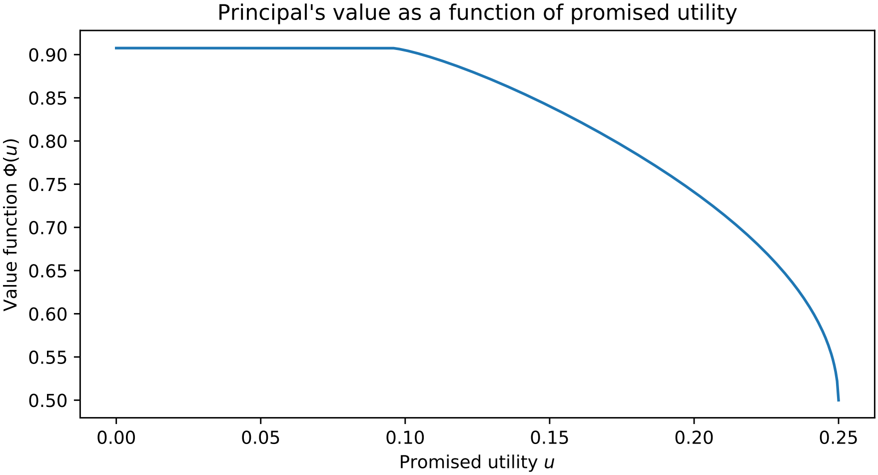
Example - Equally Patient Principal and Agent- Dynamics

$v \sim \text{Unif}[-1, 1]$, $\mathcal{U} = \left[0, \frac{1}{4}\right]$, $\delta_P = 0.8 = \delta_A$



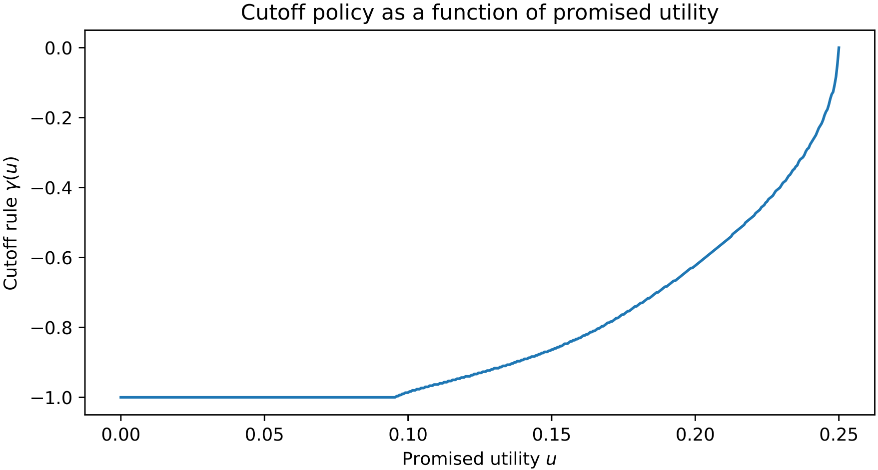
Example - Impatient Principal

$v \sim \text{Unif}[-1, 1]$, $\mathcal{U} = \left[0, \frac{1}{4}\right]$, $\delta_P = 0.7 < 0.8 = \delta_A$



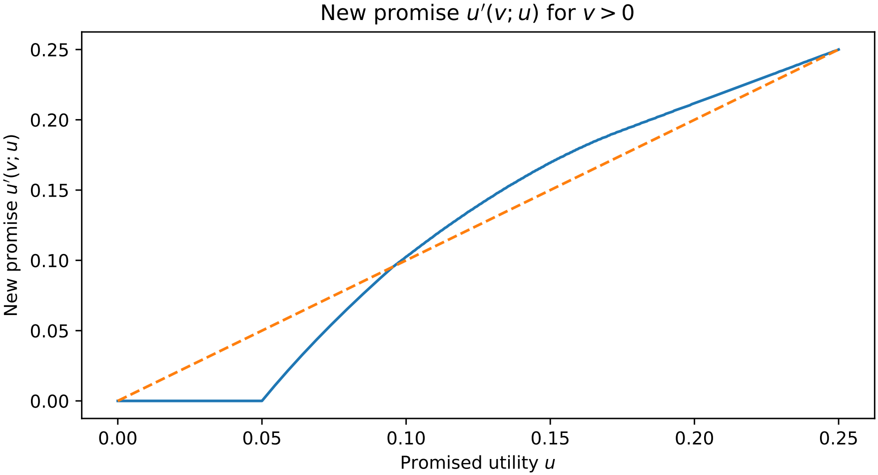
Example - Impatient Principal

$v \sim \text{Unif}[-1, 1]$, $\mathcal{U} = \left[0, \frac{1}{4}\right]$, $\delta_P = 0.7 < 0.8 = \delta_A$



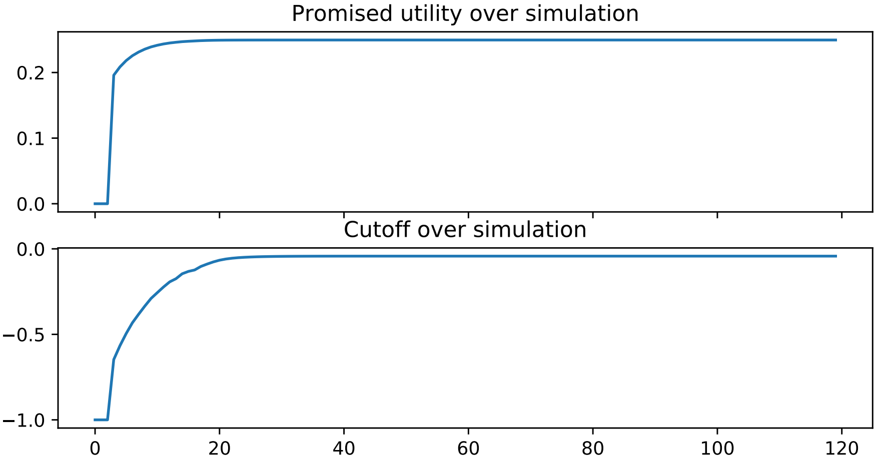
Example - Impatient Principal

$v \sim \mathrm{Unif}[-1,1]$, $\mathcal{U} = \left[0, \frac{1}{4}\right]$, $\delta_P = 0.7 < 0.8 = \delta_A$



Example - Impatient Principal - Dynamics

$v \sim \text{Unif}[-1,1]$, $\mathcal{U} = \left[0, \frac{1}{4}\right]$, $\delta_P = 0.7 < 0.8 = \delta_A$



Intuition of proof

Maximize Lagrangian for

$$\max_{x(v;u),u'(v;u)} \mathbb{E}_{v\sim F}\left[(1-\delta_P)x(v;u)+\delta_P\Phi\left(u'(v;u)\right)\right] \text{ subject to}$$

$$\mathbb{E}_{v\sim F}\left[(1-\delta_A)vx(v;u)+\delta_A u'(v;u)\right] \geq u, \quad \lambda(u) \qquad \text{(PK)}$$

$$(1-\delta_A)vx(v;u)+\delta_A u'(v;u) \geq 0, \text{ for each } v. \quad \mu(v;u) \qquad \text{(PC)}$$

First-order conditions:

- $x(v;u)=1$ iff $v > -\frac{1-\delta_P}{1-\delta_A}\frac{1}{\lambda(u)+\mu(v;u)}$.
- $\Phi'(u'(v;u)) = \frac{-\delta_A}{\delta_P}(\lambda(u)+\mu(v;u))$.

Envelope theorem: $\Phi'(u) = -\lambda(u) \implies$ where $v>0$, $\Phi'(u'(v;u)) = \frac{\delta_A}{\delta_P}\Phi'(u)$. □

Roadmap

Single agent model

Single agent optimal contract

Multiple agents

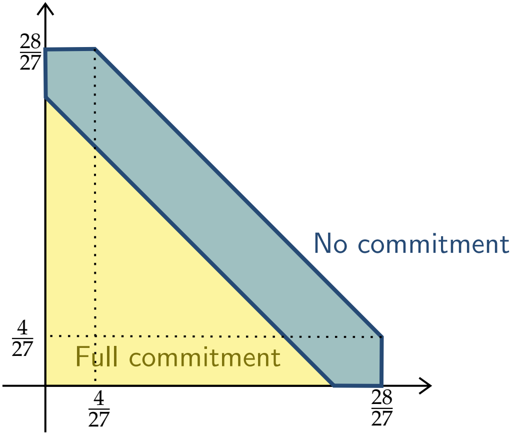
Conclusion and next steps

Multiple Agent Model

Agents and timing

- Now N agents and N indivisible items in each period.
- Principal now offers a matching $M \in \mathcal{M}(v)$ of items and agents.
- $\mathcal{U}$ is now a symmetric polytope in $\mathbb{R}_+^N$.
- Full commitment no longer necessary for results.

Example, suppose $v \sim \mathrm{Unif}[-1, 2]$



Value function properties

Schur-concavity

Existence, uniqueness, monotonicity and concavity follow as previously.

Value function properties

Schur-concavity

Existence, uniqueness, monotonicity and concavity follow as previously.

Majorization preorder: $u \prec u'$ if (after ordering components of u and u' in descending order), we have that for all k,

$$\sum_{i=1}^{k} u_i \leq \sum_{i=1}^{k} u_i', \text{ and } \sum_{i=1}^{N} u_i = \sum_{i=1}^{N} u_i'.$$

e.g. $\left(\frac{1}{n}, \ldots, \frac{1}{n}\right) \prec \left(\frac{1}{n-1}, \ldots, \frac{1}{n-1}, 0\right) \prec \cdots \prec \left(\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0\right) \prec (1, 0, \ldots, 0)$.

Value function properties

Schur-concavity

Existence, uniqueness, monotonicity and concavity follow as previously.

Majorization preorder: $u \prec u'$ if (after ordering components of u and u' in descending order), we have that for all k,

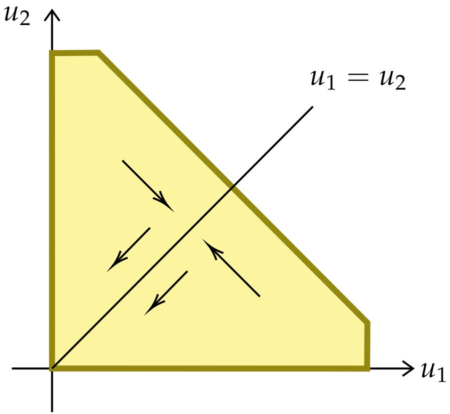
$$\sum_{i=1}^{k} u_i \leq \sum_{i=1}^{k} u'_i, \text{ and } \sum_{i=1}^{N} u_i = \sum_{i=1}^{N} u'_i.$$

e.g. $\left(\frac{1}{n},\ldots,\frac{1}{n}\right) \prec \left(\frac{1}{n-1},\ldots,\frac{1}{n-1},0\right) \prec \cdots \prec \left(\frac{1}{2},\frac{1}{2},0,\ldots,0\right) \prec (1,0,\ldots,0)$.

Symmetry + concavity $\Rightarrow$ Schur-concavity: Φ is *decreasing* in the majorization preorder.

Equalization of promised utilities

Schur-concavity of Φ implies that the principal prefers equalization of promised utilities among agents.



Implication for design: "Loyalty"

Theorem

In the optimal mechanism, the matching of items in any period is assortative in u and v.

That is, those agents with the highest promised utility ($\iff$ worst historical allocations) receive the best arriving items in any period.

Intuition: allocating better items to a worse-off agent slackens the associated promise-keeping constraint and allows the principal to equalize promised utilities in the Schur-concave objective.

Proof idea

Maximize Lagrangian for

$$\max_{M(v;u)\in\mathcal{M}(v),u'(v;u)\in\mathcal{U}} \mathbb{E}_{v\sim F}\left[(1-\delta_P)|M(v;u)| + \delta_P \Phi\left(u'(v;u)\right)\right] \text{ subject to}$$

$$\mathbb{E}_{v\sim F}\left[(1-\delta_A)v_i^M(v;u) + \delta_A u_i'(v;u)\right] \geq u_i, \text{ for each } i, \quad \lambda_i(u) \qquad \text{(PK)}$$

$$(1-\delta_A)v_i^M(v;u) + \delta_A u_i'(v;u) \geq 0, \text{ for each } i \text{ and } v. \quad \mu_i(v;u) \qquad \text{(PC)}$$

Proof idea

Maximize Lagrangian for

$$\max_{M(v;u)\in\mathcal{M}(v),u'(v;u)\in\mathcal{U}} \mathbb{E}_{v\sim F}\left[(1-\delta_P)|M(v;u)| + \delta_P\Phi\left(u'(v;u)\right)\right] \text{ subject to}$$

$$\mathbb{E}_{v\sim F}\left[(1-\delta_A)v_i^M(v;u) + \delta_A u_i'(v;u)\right] \geq u_i, \text{ for each } i, \quad \lambda_i(u) \qquad \text{(PK)}$$

$$(1-\delta_A)v_i^M(v;u) + \delta_A u_i'(v;u) \geq 0, \text{ for each } i \text{ and } v. \quad \mu_i(v;u) \qquad \text{(PC)}$$

Optimality for M and envelope theorem:

$$M(v;u) \text{ solves } \max_{M\in\mathcal{M}(v)} (1-\delta_P)|M| + (1-\delta_A)\lambda(u)\cdot v_i^M + (1-\delta_A)\mu(v;u)\cdot v_i^M$$

$$\nabla\Phi(u) = -\lambda(u).$$

Proof idea

For simplicity, consider $v \gg 0$, so that $\mu(v;u) = 0$.

Then

$$M(v;u) \text{ solves } \max_{M \in \mathcal{M}(v)} (1-\delta_P)|M| - (1-\delta_A)\nabla\Phi(u) \cdot v_i^M.$$

Proof idea

For simplicity, consider $v \gg 0$, so that $\mu(v;u) = 0$.

Then

$$M(v;u) \text{ solves } \max_{M \in \mathcal{M}(v)} (1-\delta_P)|M| - (1-\delta_A)\nabla\Phi(u) \cdot v_i^M.$$

Schur-Ostrowski criterion for Schur-concave functions:

$$(u_i - u_j)\left(\frac{\partial \Phi}{\partial u_i} - \frac{\partial \Phi}{\partial u_j}\right) \leq 0, \text{ i.e. } u_i < u_j \implies \frac{\partial \Phi}{\partial u_i} < \frac{\partial \Phi}{\partial u_j} (\leq 0).$$

Proof idea

For simplicity, consider $v \gg 0$, so that $\mu(v;u) = 0$.

Then

$$M(v;u) \text{ solves } \max_{M \in \mathcal{M}(v)} (1-\delta_P)|M| - (1-\delta_A)\nabla\Phi(u) \cdot v_i^M.$$

Schur-Ostrowski criterion for Schur-concave functions:

$$(u_i - u_j)\left(\frac{\partial\Phi}{\partial u_i} - \frac{\partial\Phi}{\partial u_j}\right) \leq 0, \text{ i.e. } u_i < u_j \implies \frac{\partial\Phi}{\partial u_i} < \frac{\partial\Phi}{\partial u_j} (\leq 0).$$

So larger v_i^M should be paired with larger $-\frac{\partial\Phi}{\partial u_i} \implies$ assortativity. □

Roadmap

Single agent model

Single agent optimal contract

Multiple agents

Conclusion and next steps

Conclusion and next steps

I introduce a simple model of dynamic allocation and matching over time.

In the optimal contract, the principal promises better future allocations to incentivize the agent to accept disliked allocations today.

The principal rewards "loyalty" by prioritizing agents with worse historical allocations for better allocations today.

Implication: Suggests that the first-come-first-serve mechanism used by many rideshare platforms may be suboptimal.

Next steps: fuller characterization and simulation of $N \geq 2$, stochastic arrival of agents
Speculative next steps: price benchmark, unobservable heterogeneity in values

Thank you!

Matching teachers in Queensland

	Year 1	Year 2	Year 3	Year 4	Year 5
Rating 7	11	11	17	20	22
Rating 6	7	7	7	10	12
Rating 5	5	5	5	7	8
Rating 4	4	4	4	6	7
Rating 3	3	3	3	4	5
Rating 2	2	2	2	2	2
Rating 1	1	1	1	1	1

- For each year of teaching, a teacher earns ‘transfer points’.
- Less desirable schools earn more transfer points.
- At start of school year, a teacher may apply to vacant jobs in schools.
- Priority given to teachers with highest transfer points balance.

Back to Main