Topping Up and Optimal Redistribution

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Motivation

Governments **redistribute** by subsidising consumption:

- **food assistance** (e.g. SNAP);
- **housing** (e.g. public housing programs, LIHTC);
- other examples: childcare, education, transport, health care, energy.

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Topping up can also be a **design choice** of subsidy programs:

- Implementation through vouchers or direct provision?
- Explicit banning? e.g. housing loan subsidies that depend on apartment size.

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$\leadsto$ Topping up weakens the **screening value of low quantities**.

Research question and approach

> *“Programs with or without the possibility of topping up have different welfare properties. Currently, there are no general results regarding the merits of the two. [...] Nor are there any general results on the characterization of optimal public provision policies, targeted or universal.”*
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Approach: Study a nonlinear in-kind subsidy for a good also supplied by a private market.

- Types differ in demand θ and welfare weight $\omega(\theta)$.
- The planner faces opportunity cost α of public funds.
- Cash transfers are restricted, so redistribution occurs solely through the subsidy schedule.

This paper

Private market access changes the feasible price schedules:

- **Topping Up**: subsidised marginal prices cannot exceed the private market price.
- **No Topping Up**: subsidised average prices cannot exceed the private market price.

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Main results:

- characterise when topping up affects the **scope** for subsidies; and
- characterise how topping up affects the **optimal design** of subsidies.

Related work

- **Redistributive Mechanism Design.** Weitzman (1977), Che, Gale and Kim (2013), Condorelli (2013), Dworczak, Kominers and Akbarpour (2021), Akbarpour, Dworczak and Kominers (2022, 2024), Kang (2023, 2024), Akbarpour, Budish, Dworczak and Kominers (2024), Pai and Strack (2024).
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- **"Partial" Mechanism Design.** Philippon and Skreta (2012), Tirole (2012), Fuchs and Skrzypacz (2015), Dworczak (2020), Loertscher and Muir (2022), Loertscher and Marx (2022), Kang and Muir (2022), Kang (2023).
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- **Methodological Tools in Mechanism Design.**
  - Generalised ironing: Toikka (2011).
  - Lagrangian approach: Amador, Werning and Angeletos (2006), Amador and Bagwell (2013).
  - Type-dependent outside options: Jullien (2000), Dworczak and Muir (2024), Corrao, Flynn and Sastry (2023), Valenzuela-Stookey and Poggi (2024).
  - Majorisation and convex-order tools: Kleiner, Moldovanu and Strack (2021, 2026), Yang and Zentefis (2024).
  - ⇝ **This paper:** majorisation constraints due to type-dependent outside options in a convex program.

Related work

- **Public Finance.**
  Theory: Ramsey (1927), Diamond (1975), Mirrlees (1976, 1986), Atkinson and Stiglitz (1976), Nichols and Zeckhauser (1982), Hammond (1987), Blackorby and Donaldson (1988), Coate (1989), Besley and Coate (1991), Coate, Johnson and Zeckhauser (1994), Blomquist and Christiansen (1998), Gahvari and Mattos (2007).

  Empirics: Baum-Snow and Marion (2009), Diamond and McQuade (2019), van Dijk (2019), Dinerstein, Neilson and Otero (2020), Atal, Cuesta, Gonzalez and Otero (2021), Dinerstein and Smith (2021), Jimenez-Hernandez and Seira (2021), Handbury and Moshary (2021).

  $\rightsquigarrow$ **This paper:** consumer heterogeneity in consumption preferences that are correlated with incomes.

  - Atkinson and Stiglitz (1976) theorem fails with preference heterogeneity: Saez (2002), Pai and Strack (2024).
  - Related recent theory: Doligalski, Dworczak, Akbarpour and Kominers (2025), Doligalski, Dworczak, Krysta and Tokarski (2025).

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  As Doligalski, Dworczak, Akbarpour and Kominers (2025) write:

  "Several decades after the original work of Atkinson and Stiglitz (1976), conventional economic wisdom seems to have embraced the idea of using income taxes rather than goods market interventions to redistribute; it would appear that this intuition needs to be revisited, and more research is needed to understand whether it constitutes good policy advice under realistic scenarios."

Model

Setup

Consumers:

- There is a unit mass of risk-neutral consumers in a market for a divisible, homogeneous good.
- Consumers differ in type $\theta \in [\underline{\theta}, \overline{\theta}]$ with $\underline{\theta} \geq 0$, and $\theta \sim F$, continuous with density $f > 0$.
- Each consumer derives utility $\theta v(q) - t$ from quantity/quality $q \in [0, A]$ given payment t.
  $v : [0, A] \to \mathbb{R}$ is differentiable with $v' > 0$, $v'' < 0$ and $v' \to \infty$ as $q \downarrow 0$.

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Extensions (not today): equilibrium effects, observable characteristics, product choice and eligibility.

Laissez-faire equilibrium

- Perfectly competitive private market $\rightsquigarrow$ laissez-faire price $p^{\mathrm{LF}} = c$ per unit.
- Each consumer solves

$$U^{\mathrm{LF}}(\theta) := \max_{q \in [0,A]} \left[\theta v(q) - cq\right].$$

v is strictly concave $\rightsquigarrow$ unique maximiser:

$$q^{\mathrm{LF}}(\theta) = (v')^{-1}\left(\frac{c}{\theta}\right) = D(c, \theta).$$

- To simplify statements of some results, assume today that $q^{\mathrm{LF}}(\underline{\theta}) > 0$.

Subsidy design

The social planner contracts costlessly with firms and sells using a **subsidised payment schedule** $P^{\sigma}(q)$.

$\rightsquigarrow \Sigma(q) = cq - P^{\sigma}(q)$ is the **total subsidy** as a function of q, and $\sigma(q) = \Sigma'(q)$ is the **marginal subsidy**.

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Key assumption: The social planner can subsidise consumption but **not make lump-sum cash transfers**,

$$\rightsquigarrow \; P^{\sigma}(q) \geq 0 \text{ for all } q.$$

In-kind restriction

Taxation

Private market interaction

No Topping Up: given a price schedule $P(q)$, the effective price schedule is the lower envelope $P^{\mathrm{eff}}(q) = \min\{P(q), cq\}$.

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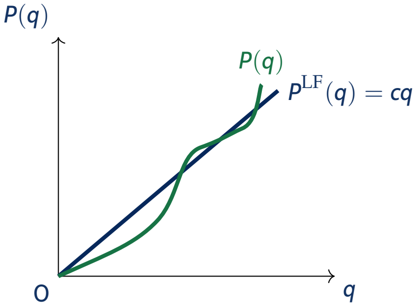

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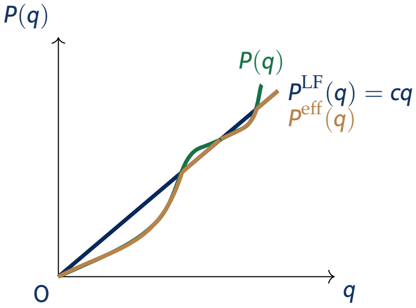

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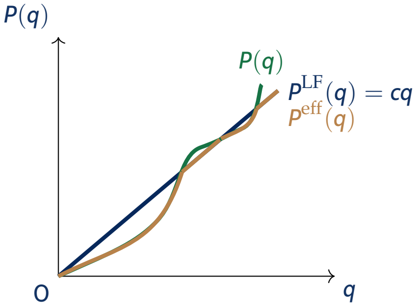

Direct mechanism: $P(q) \leq cq$

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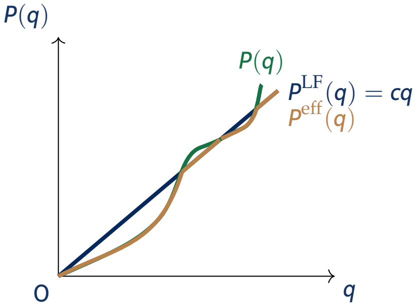

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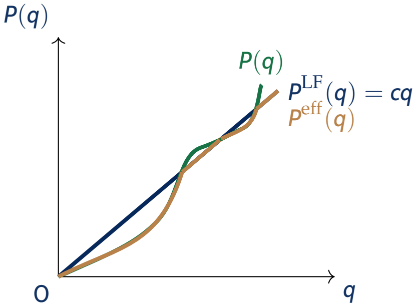

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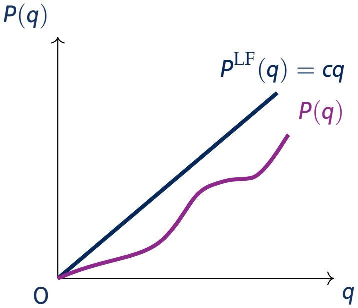

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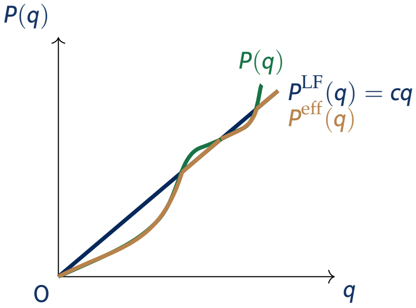

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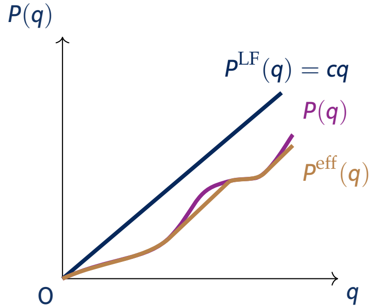

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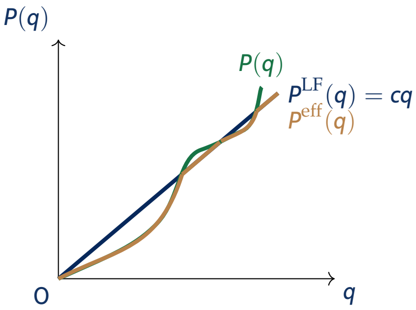

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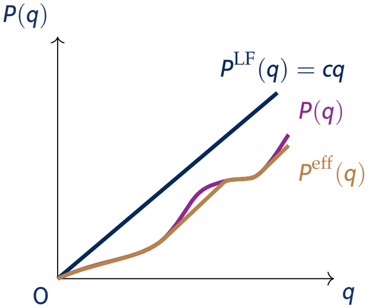

Direct mechanism: $P'(q) \leq c$

$\iff q(\theta) \geq q^{\mathrm{LF}}(\theta)$

$\iff \Sigma(q)$ is increasing in q.

Implementation proof

Redistributive objective

The social planner seeks to maximise **weighted total surplus**.

- **Consumer surplus**: the social planner assigns a welfare weight $\omega(\theta)(:= \mathbf{E}[\omega|\theta])$ to consumer type θ.
  $\rightsquigarrow$ $\omega(\theta)$, expected social value of giving consumer θ one unit of money, continuous in θ.

▸ Endogenising ω and α

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$\rightsquigarrow$ Objective:

$$\max_{P^\sigma(q)\geq 0} \int_\theta \left[\omega(\theta)U^\sigma(\theta) - \alpha\Sigma(q^\sigma(\theta))\right] \mathrm{d}F(\theta) \text{ s.t. } \underbrace{\sigma(q) \geq 0}_{\text{topping up}} \text{ or } \underbrace{\Sigma(q) \geq 0}_{\text{no topping up}}$$

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Remarks:

- If $\omega(\theta) > \alpha$, the social planner would want to transfer a dollar to type θ.
- If $\mathbf{E}_{\theta}[\omega(\theta)] > \alpha$, the social planner would want to make a lump-sum cash transfer to all consumers.

▸ Endogenising ω and α

When to subsidise

(And when not to)

When to subsidise: scope

Theorem 1. The planner can improve over laissez-faire if and only if the corresponding condition holds:

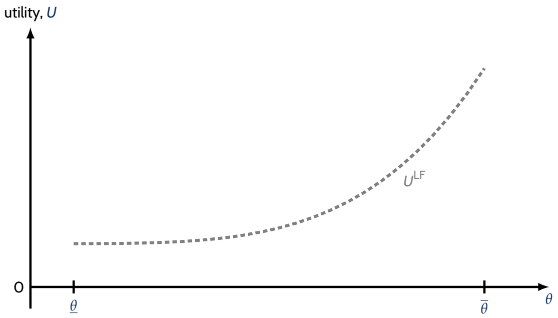
$$\textbf{No Topping Up} \quad : \quad \exists\, \hat{\theta} \in \Theta \ \text{ such that } \ \omega(\hat{\theta}) > \alpha,$$

$$\textbf{Topping Up} \quad : \quad \exists\, \hat{\theta} \in \Theta \ \text{ such that } \ \mathbf{E}_{\theta}[\omega(\theta) \mid \theta \geq \hat{\theta}] > \alpha.$$

Remark. The relevant statistic changes from the **largest welfare weight** to the **largest upper-tail average**.

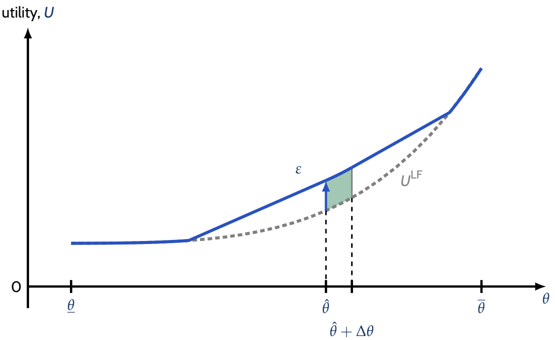
$\rightsquigarrow$ The scope for subsidies is weakly larger with **no topping up** than with **topping up**.

Visual proof: without topping up



Begin with laissez-faire utility U^{LF}.

Visual proof: without topping up



Give a subsidy ε to an interval of types with $\omega(\theta) > \alpha$.

Their consumption is unchanged, giving a positive first-order redistributive gain $\int_{\hat{\theta}}^{\hat{\theta}+\Delta\theta} (\omega(s) - \alpha)\varepsilon dF(s)$.

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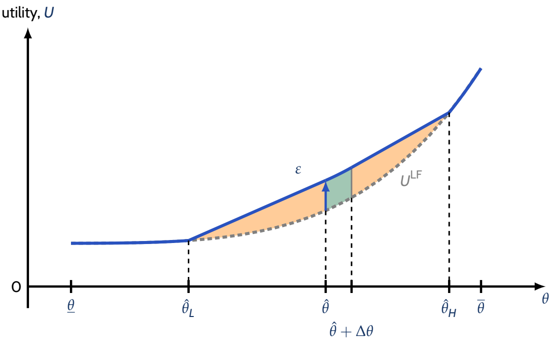

The mass of consumers distorting their consumption to obtain the subsidy is $\mathcal{O}(\varepsilon^{1/2})$.
The planner's loss per affected consumer is at most $\mathcal{O}(\varepsilon)$.

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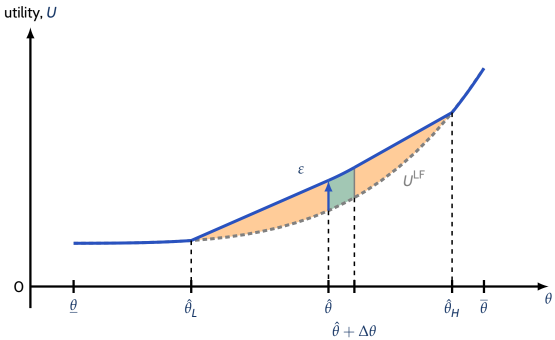

The mass of consumers distorting their consumption to obtain the subsidy is $\mathcal{O}(\varepsilon^{1/2})$.
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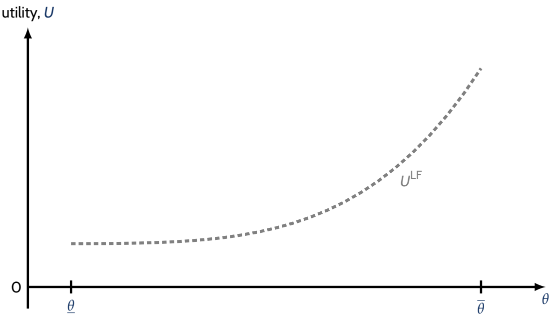
Total loss is $\mathcal{O}(\varepsilon^{3/2})$, so the first-order gain dominates for small ε.

▸ Mathematical proof

Visual proof: with topping up



Begin with laissez-faire utility U^{LF}.

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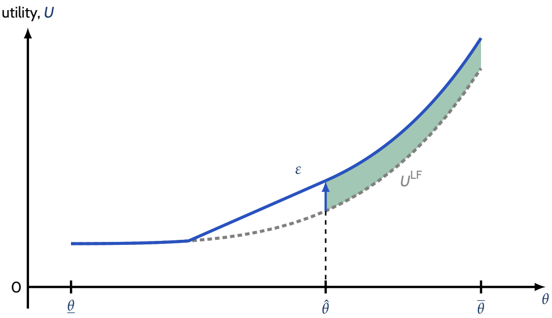

Give type $\hat{\theta}$ a subsidy ε. All higher types receive the subsidy and top up to laissez-faire consumption.

$\rightsquigarrow$ a positive first-order gain iff $\mathbf{E}[\omega(\theta) \mid \theta \geq \hat{\theta}] > \alpha$.

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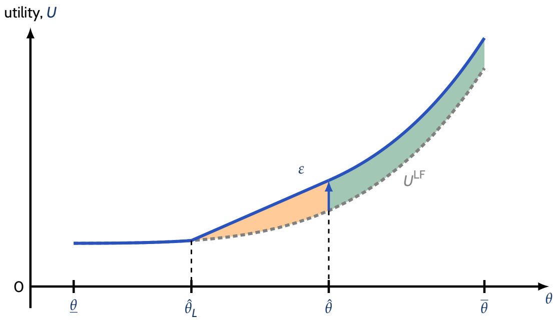
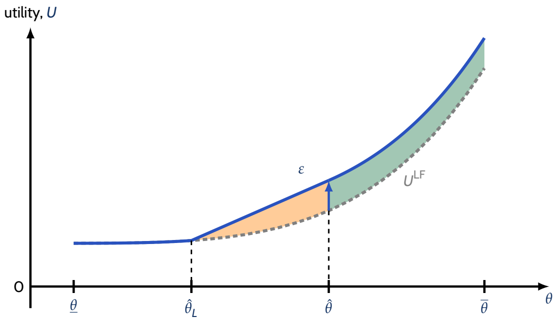


Again, the mass of consumers changing consumption is $\mathcal{O}(\varepsilon^{1/2})$.
The planner's loss per affected consumer is at most $\mathcal{O}(\varepsilon)$.

Visual proof: with topping up



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When to subsidise: economic implications

Two baseline cases:

"Negative Correlation": $\omega(\theta)$ is decreasing in θ.

- high-demand consumers tend to have less need for redistribution.
- *e.g.* food, education, and, if $\omega \propto 1/\text{Income}$, normal goods.

$\rightsquigarrow$ Wider scope for subsidies without topping up ($\omega(\underline{\theta}) > \alpha$) than with topping up ($\mathbf{E}[\omega] > \alpha$).

"Positive Correlation": $\omega(\theta)$ is increasing in θ.

- high-demand consumers tend to have greater need for redistribution.
- *e.g.* staple foods, public transport, and, if $\omega \propto 1/\text{Income}$, inferior goods.

$\rightsquigarrow$ Scope for subsidies is the same with and without topping up ($\omega(\overline{\theta}) > \alpha$).

How to subsidise

Solving the mechanism design problem

Mechanism design problem

The social planner chooses the **total allocation function** q and the **total payment function** t to maximise weighted total surplus:

$$\max_{(q,t)} \int_{\underline{\theta}}^{\overline{\theta}} \left[\omega(\theta) \underbrace{[\theta v(q(\theta)) - t(\theta)]}_{\text{consumer surplus}} - \alpha \underbrace{[cq(\theta) - t(\theta)]}_{\text{net cost}} \right] \mathrm{d}F(\theta),$$

subject to

- incentive compatibility, $\qquad \theta \in \arg\max_{\hat{\theta} \in [\underline{\theta}, \overline{\theta}]} \left[\theta v(q(\hat{\theta})) - t(\hat{\theta}) \right] \quad \forall\, \theta \in [\underline{\theta}, \overline{\theta}];$ (IC)
- no lump-sum transfers, $\qquad t(\theta) \geq 0 \quad \forall\, \theta \in [\underline{\theta}, \overline{\theta}];$ (LS)
- individual rationality, $\qquad \theta v(q(\theta)) - t(\theta) \geq U^{\mathrm{LF}}(\theta) \quad \forall\, \theta \in [\underline{\theta}, \overline{\theta}],$ (IR)
- topping-up constraint, $\qquad q(\theta) \geq q^{\mathrm{LF}}(\theta) \quad \forall \theta \in [\underline{\theta}, \overline{\theta}].$ (TU)

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subject to (IC), (LS), (IR), and (TU).

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subject to (IC), (LS), (IR), and (TU).

#1. Apply the Myerson (1981) lemma and the Milgrom and Segal (2002) envelope theorem to express the objective in terms of $\underline{U} := U(\underline{\theta})$ and a nondecreasing $q(\theta)$, substituting

$$t(\theta) = \theta v(q(\theta)) - \int_{\underline{\theta}}^{\theta} v(q(s))\,\mathrm{d}s - \underline{U}.$$

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$$\max_{\underline{U},\, q \text{ non-decreasing}} \left\{ \left[\mathbf{E}[\omega] - \alpha\right] \underline{U} + \int_{\underline{\theta}}^{\overline{\theta}} \left[\left[\alpha\theta + \frac{\int_{\theta}^{\overline{\theta}} \left[\omega(s) - \alpha\right] \, \mathrm{d}F(s)}{f(\theta)} \right] v(q(\theta)) - \alpha c q(\theta) \right] \, \mathrm{d}F(\theta) \right\},$$

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subject to (LS), (IR), and (TU).

#2. It suffices to enforce (LS) only for the lowest type $\underline{\theta}$ because $t(\theta)$ is nondecreasing by (IC), so

$$\underline{U} \leq \underline{\theta} v(q(\underline{\theta})),$$

while (IR) for $\underline{\theta}$ implies

$$\underline{U} \geq U^{\mathrm{LF}}(\underline{\theta}).$$

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subject to (IR) and (TU).

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subject to (IR) and (TU).

#3. Write the virtual type as

$$J(\theta) = \underbrace{\theta}_{\text{efficiency}} + \underbrace{\frac{\int_{\theta}^{\overline{\theta}} [\omega(s) - \alpha] \mathrm{d}F(s)}{\alpha f(\theta)}}_{\text{redistributive motive}}$$

Call $J(\theta) - \theta$ the **distortion term**. Its sign depends on $\int_{\theta}^{\overline{\theta}} \omega(s) - \alpha \, \mathrm{d}F(s)$.

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$$\max_{\substack{U^{\mathrm{LF}}(\underline{\theta}) \leq \underline{U} \leq \underline{\theta} v(q(\underline{\theta})), \\ q \text{ non-decreasing}}} \left[\mathbf{E}[\omega] - \alpha\right] \underline{U} + \alpha \int_{\underline{\theta}}^{\overline{\theta}} \underbrace{\left[J(\theta) v(q(\theta)) - c q(\theta)\right]}_{\text{surplus of virtual type}} \, \mathrm{d}F(\theta),$$

subject to (IR) and (TU).

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subject to (IR) and (TU).

#4. By the envelope theorem, (TU) and (IR) for $\underline{\theta}$ imply (IR) for all θ.

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subject to
with topping up:

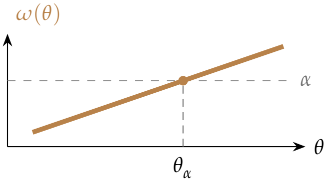
$$q(\theta) \geq q^{\mathrm{LF}}(\theta) \qquad \forall\, \theta \in [\underline{\theta}, \overline{\theta}], \tag{FOSD}$$

without topping up:

$$\underline{U} + \int_{\underline{\theta}}^{\theta} v(q(s)) \mathrm{d}s \geq U^{\mathrm{LF}}(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} v(q^{\mathrm{LF}}(s)) \mathrm{d}s, \qquad \forall\, \theta \in [\underline{\theta}, \overline{\theta}], \tag{SOSD}$$

Optimal design with positive correlation

Suppose $\omega(\theta)$ is increasing in θ (e.g. public transport, staple goods).



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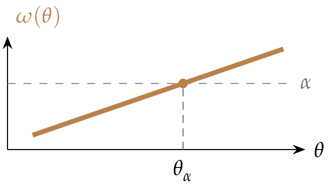

$$J(\theta) = \theta + \frac{\int_\theta^{\overline{\theta}} [\omega(s) - \alpha] \, dF(s)}{\alpha f(\theta)}.$$

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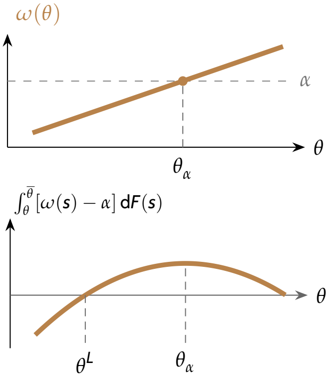

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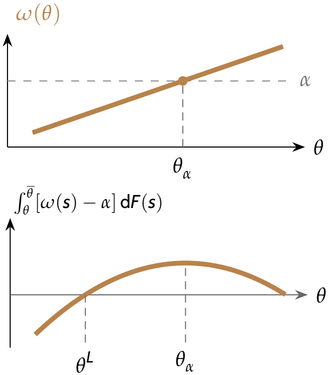

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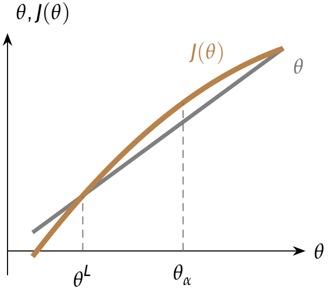

J crosses θ at most once **from below**: zero or one interior crossing.

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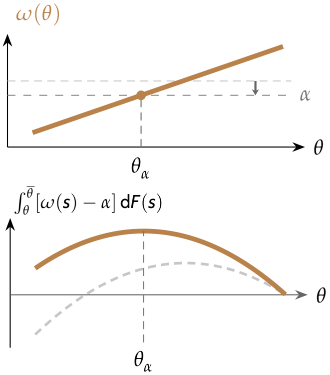

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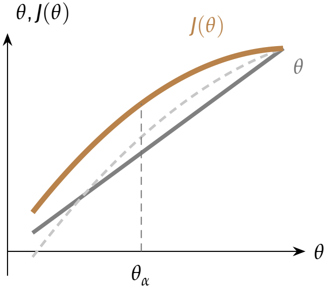


J crosses θ at most once **from below**: zero or one interior crossing.
Lowering α raises the tail integral and J; here $J > \theta$ throughout the interior.

Optimal design with positive correlation

$$\max_{\substack{U^{\mathrm{LF}}(\underline{\theta})\leq \underline{U}\leq \underline{\theta} v(q(\underline{\theta})),\\ q \text{ non-decreasing}}} \left[\mathbf{E}[\omega]-\alpha\right]\underline{U}+\alpha\int_{\underline{\theta}}^{\overline{\theta}}\left[J(\theta)v(q(\theta))-cq(\theta)\right]\mathrm{d}F(\theta) \quad \text{s.t.} \quad \begin{array}{l} U(\theta)\geq U^{\mathrm{LF}}(\theta) \\ q(\theta)\geq q^{\mathrm{LF}}(\theta) \end{array}$$

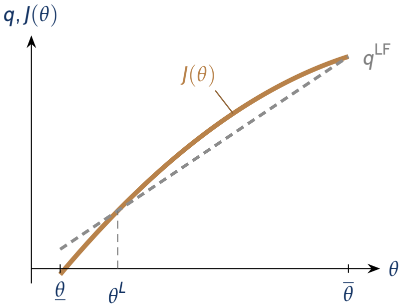
The optima **with topping up** and **without topping up** coincide.

If $\mathbf{E}[\omega]<\alpha<\max\omega$, then

$$q^*(\theta)=\begin{cases} q^{\mathrm{LF}}(\theta) & \text{for } \theta\leq\theta^L, \\ D(c,\bar{J}(\theta)) & \text{for } \theta\geq\theta^L, \end{cases}$$

where $\mathbf{E}[\omega(\theta)\mid\theta\geq\theta^L]=\alpha$.

NB: Figures use $v(q)=c\log q \rightsquigarrow D(c,z)=z$.



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$$\max_{\substack{U^{\mathrm{LF}}(\underline{\theta}) \le \underline{U} \le \underline{\theta} v(q(\underline{\theta})),\\ q \text{ non-decreasing}}} [\mathbf{E}[\omega] - \alpha]\,\underline{U} + \alpha \int_{\underline{\theta}}^{\overline{\theta}} [J(\theta) v(q(\theta)) - c q(\theta)]\,\mathrm{d}F(\theta) \quad \text{s.t.} \quad \begin{array}{l} U(\theta) \ge U^{\mathrm{LF}}(\theta) \\ q(\theta) \ge q^{\mathrm{LF}}(\theta) \end{array}$$

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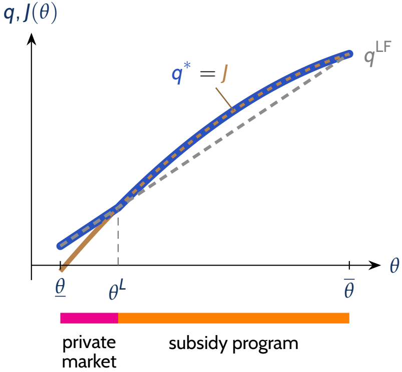

▸ Numerical example

Optimal design with positive correlation

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If $\mathbf{E}[\omega]>\alpha$, then

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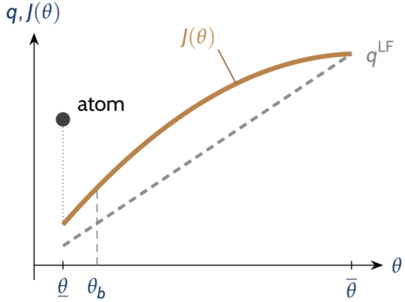

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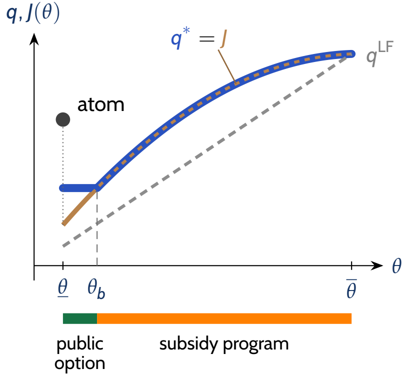

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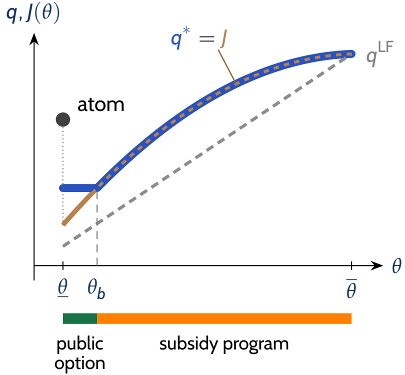

Intuition: the planner's objective decreases with divergence from J.

▸ Numerical example

(Proof idea)

Positive correlation

For $\mathbf{E}[\omega] < \alpha < \max \omega$, let $\lambda(\theta)$ be the (IR) multiplier in the **NTU** problem.

$$\mathcal{L}(q, \underline{U}, \lambda) = \ldots + \int_{\underline{\theta}}^{\overline{\theta}} \lambda(\theta) [U(\theta) - U^{\mathsf{LF}}(\theta)] \, \mathrm{d}F(\theta)$$

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Integrating by parts and applying the envelope theorem, rewrite the Lagrangian using $\Lambda(\theta) = \int_{\theta}^{\overline{\theta}} \lambda(s) \, \mathrm{d}F(s)$ as

$$\mathcal{L}(q, \underline{U}, \lambda) = \ldots + \int_{\underline{\theta}}^{\overline{\theta}} \frac{\Lambda(\theta)}{f(\theta)} [v(q(\theta)) - v(q^{\mathrm{LF}}(\theta))] \, \mathrm{d}F(\theta)$$

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Complementary slackness $\rightsquigarrow \Lambda(\theta)$ is nonnegative and nonincreasing in θ, constant where IR is non-binding, and

$$q(\theta) = D\left(c, \overline{J + \frac{\Lambda}{\alpha f}}(\theta)\right).$$

(Proof idea)

Ironing details

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Guess and check strong convex duality for Lagrange multipliers $\Lambda(\theta) = 0$ for $\theta > \theta^L$ and $\Lambda(\theta) = -\int_{\theta}^{\overline{\theta}} [\omega(s) - \alpha]\, \mathrm{d}F(s)$ for $\theta \leq \theta^L$ (nonnegative and nonincreasing), equivalent to $q = q^*$. $\square$

Economic implications

With **positive correlation** between ω and θ:

\# 1. The social planner derives **no benefit** from restricting topping up in the private market.

\# 2. Optimal subsidies are **self-targeting**, with benefits flowing only to consumers with higher need.

\# 3. The social planner **prefers subsidies** to lump-sum cash transfers.

When do the optimal mechanisms coincide?

Roughly, positive correlation is the only case in which the **topping up** and **no topping up** problems coincide.

More formally, let θ^L be the first type at which the upper-tail integral of $\omega - \alpha$ is nonnegative.

Theorem 2. The optimal mechanisms coincide exactly when:

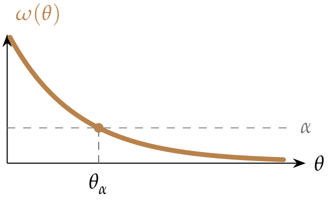
1. Below θ^L, every type has $\omega(\theta) \leq \alpha$.

2. At and above θ^L, the optimal construction without topping up already delivers at least laissez-faire consumption.

▸ Formal condition

Optimal subsidy design with negative correlation

Suppose $\omega(\theta)$ is decreasing in θ (e.g. normal goods with income-sensitive demand, private transport).



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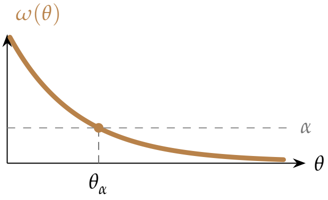

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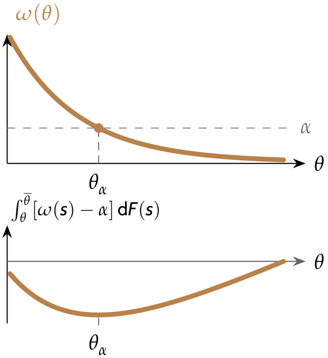

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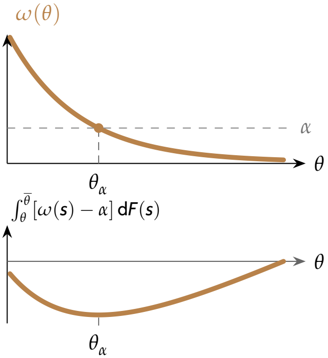

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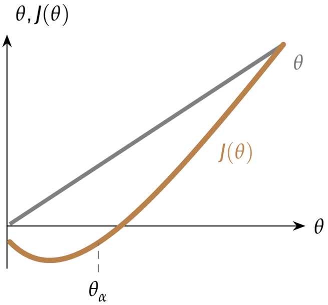

J crosses θ at most once **from above**: zero or one interior crossing.

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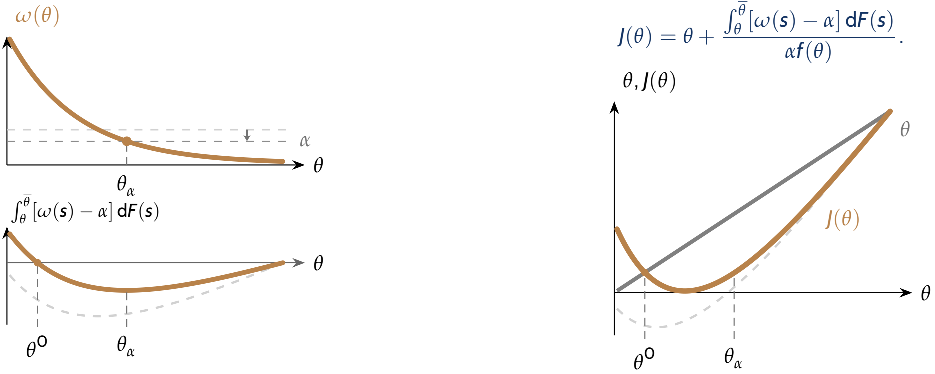

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Lowering α raises the tail integral and J; here an interior crossing appears.

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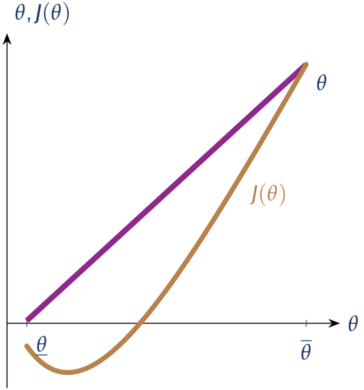
With negative correlation and $\mathbf{E}[\omega] \leq \alpha < \max \omega$, **topping up** gives laissez-faire. The planner intervenes with **no topping up**.



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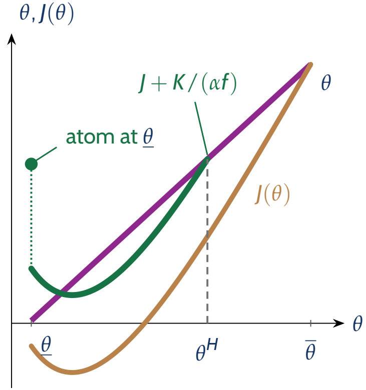
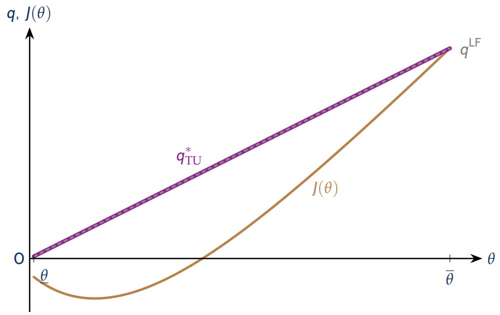


No Topping Up:

- Below θ^H, the cumulative participation multiplier is constant: $\Lambda(\theta) = K$.
- The adjusted virtual type is $J + K/(\alpha f)$. When f is constant, this is a constant upward shift.
- Include the bottom atom if $\mu > 0$; iron and apply $D(c, \cdot)$ to obtain q^*.
- Choose the (LS) multiplier μ and cutoff θ^H jointly so (IR) binds there; $K = \mu + \alpha - \mathbf{E}[\omega]$.
- Above θ^H, $q^* = q^{\text{LF}}$ and $U^* = U^{\text{LF}}$.

Negative correlation: topping up can prevent intervention

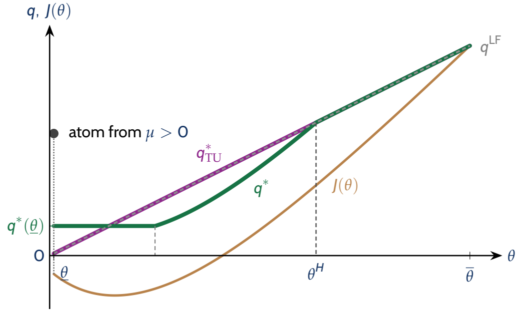
$$\mathbf{E}[\omega] \leq \alpha < \max \omega.$$



With topping up, the optimum is laissez-faire.

Negative correlation: topping up can prevent intervention

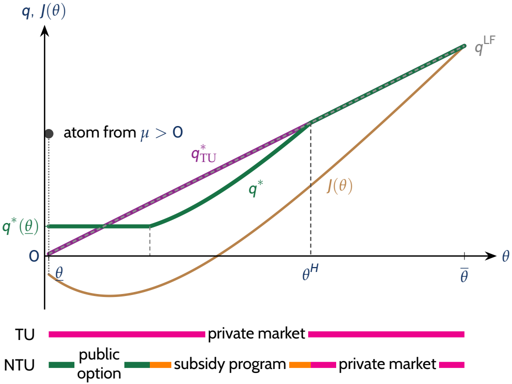
$$\mathbf{E}[\omega] \leq \alpha < \max \omega.$$



Without topping up, the planner can target low types.

Negative correlation: topping up can prevent intervention

$$\mathbf{E}[\omega] \leq \alpha < \max \omega.$$

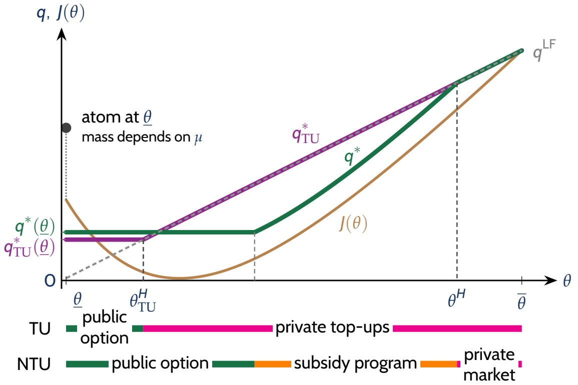


A free public option may arise; the figure illustrates a mechanism with one.

Numerical example

Negative correlation: comparing the optimal allocations

$\alpha_{\min} < \alpha < \mathbf{E}[\omega]$: both mechanisms intervene.



Without topping up, lower quantities let the planner phase out subsidies.

▸ Topping-up details

▸ No-topping-up details

▸ Numerical example

Negative correlation: economic implications

Proposition 3. With negative correlation, compare **topping up** with **no topping up**:

1. **Consumption.** The allocations cross at most once: lower types receive weakly less consumption, and higher types weakly more.

2. **Public provision.** Topping up weakly reduces the scope for a free public option.

3. **Upper cutoff.** Topping up weakly lowers the highest type whose total consumption differs from laissez-faire.

Above the topping-up cutoff, consumers can still retain an inframarginal subsidy while buying extra units privately.

▸ Exact allocation comparison

Discussion

Differences in practice

When? With topping up, the scope for intervention is larger with positive correlation ($\max \omega > \alpha$) than with negative correlation ($\mathbf{E}[\omega] > \alpha$).

In practice, many government programs focus on goods consumed disproportionately by the needy (e.g. *Tamween* bread, Indonesian fuel subsidies, dental subsidies).

Differences in practice

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In practice, many government programs focus on goods consumed disproportionately by the needy (e.g. *Tamween* bread, Indonesian fuel subsidies, dental subsidies).

How? Significant differences in marginal subsidy schedules are observed in practice:

Larger subsidies for low q

- Food stamps (SNAP)
- Women, Infants & Children (WIC) Program
- Housing Choice (Section 8) Vouchers
- Lifeline (Telecomm. Assistance) Program
- Public Housing Programs (no topping up)

Larger subsidies for high q

- Public transport fare capping
- Pharmaceutical subsidy programs
- Government-subsidised childcare places

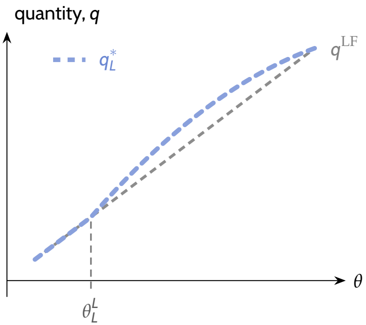
Comparative statics: higher welfare weights

Positive correlation

Proposition 1. Let increasing ω_H, ω_L satisfy $\omega_H \geq \omega_L$ pointwise. Hold F, v, c, α fixed.

Both mechanisms

- Virtual values rise at every type.
- Consumption weakly rises at every type: $q_H^*(\theta) \geq q_L^*(\theta)$.
- The program expands towards lower types: $\theta_H^L \leq \theta_L^L$.
- The (LS) multiplier weakly rises: $\mu_H^* \geq \mu_L^*$.



Exact comparisons

Intuition

Numerical example

Comparative statics: higher welfare weights

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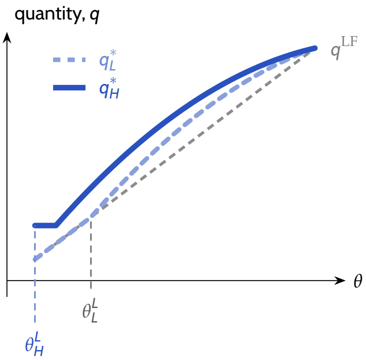


Exact comparisons

Intuition

Numerical example

Comparative statics: higher welfare weights

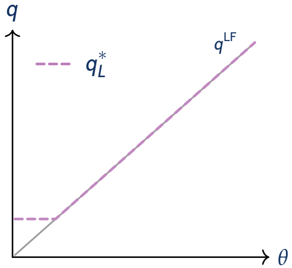
Negative correlation

Proposition 4. Let decreasing ω_H, ω_L satisfy $\omega_H \geq \omega_L$ pointwise. Hold F, v, c, α fixed.

Both optimal (LS) multipliers and both upper cutoffs weakly increase.

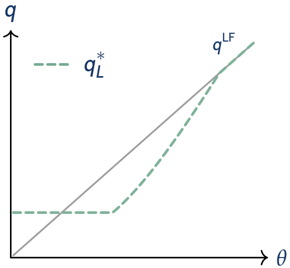
With topping up

$q^*_{\mathrm{TU},H}(\theta) \geq q^*_{\mathrm{TU},L}(\theta)$ for every θ.



Without topping up

The allocation can rotate towards low types.



NTU: there exists $\hat{\theta} \in [\underline{\theta}, \overline{\theta}]$ such that $q^*_H(\theta) \geq q^*_L(\theta)$ for $\theta < \hat{\theta}$, and $q^*_H(\theta) \leq q^*_L(\theta)$ for $\theta > \hat{\theta}$.

▸ Exact comparisons ▸ TU intuition ▸ NTU intuition ▸ Numerical example

Comparative statics: higher welfare weights

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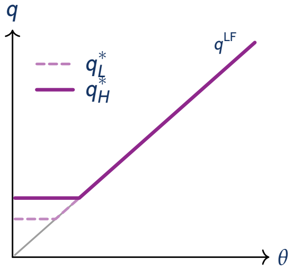

Without topping up

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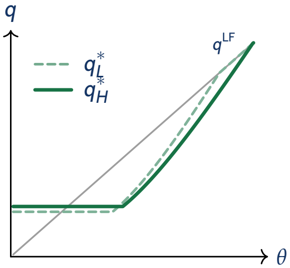

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Exact comparisons
TU intuition
NTU intuition
Numerical example

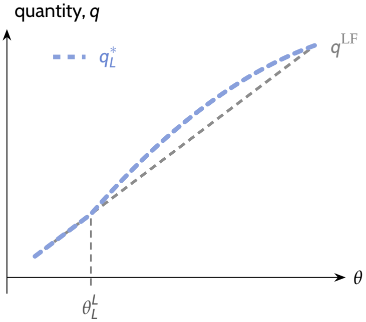
Comparative statics: a mean-preserving spread

Positive correlation

Proposition 2. Let increasing ω_H, ω_L have equal means and satisfy $\int_{\underline{\theta}}^{\overline{\theta}} [\omega_H(s) - \omega_L(s)] \, dF(s) \geq 0$ for every θ. Hold F, v, c, α fixed.

Both mechanisms

- Virtual values rise at every type.
- Consumption weakly rises at every type: $q_H^*(\theta) \geq q_L^*(\theta)$.
- The program expands towards lower types: $\theta_H^L \leq \theta_L^L$.
- The (LS) multiplier is unchanged: $\mu_H^* = \mu_L^*$.



Exact comparisons
Intuition
Numerical example

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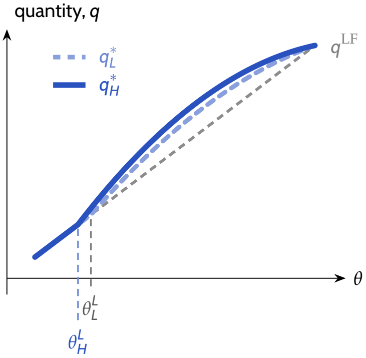


Exact comparisons

Intuition

Numerical example

Comparative statics: a mean-preserving spread

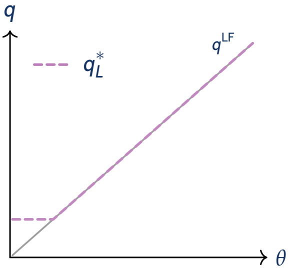
Negative correlation

Proposition 5. Let decreasing ω_H, ω_L have equal means and satisfy $\int_{\underline{\theta}}^{\theta} [\omega_H(s) - \omega_L(s)]\, dF(s) \geq 0$ for every θ. Hold F, v, c, α fixed.

With topping up

$\mu^*_{\mathrm{TU},H} = \mu^*_{\mathrm{TU},L}$; the upper cutoff weakly falls.

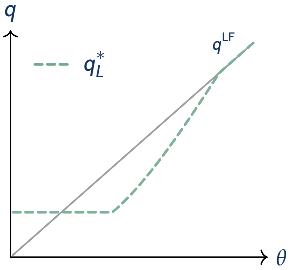
$q^*_{\mathrm{TU},H}(\theta) \leq q^*_{\mathrm{TU},L}(\theta)$ for every θ.



Without topping up

$\mu^*_H \geq \mu^*_L$.

Allocation and cutoff effects are ambiguous.



▸ Exact comparisons ▸ TU intuition ▸ NTU intuition ▸ Numerical example

Comparative statics: a mean-preserving spread

Negative correlation

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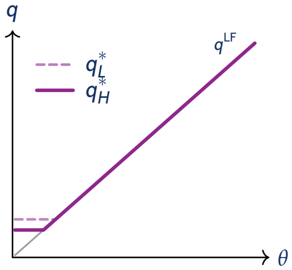

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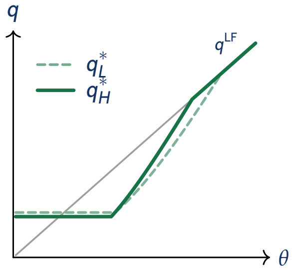

▸ Exact comparisons ▸ TU intuition ▸ NTU intuition ▸ Numerical example

Conclusion

Summing up

- **Topping up reduces the screening power of low quantities.**
- **Topping up reduces the scope of intervention.** The relevant statistic is the highest individual welfare weight **without topping up**, and the highest upper-tail average **with topping up**.
- **With negative correlation, restricting topping up can improve targeting:** e.g., public housing versus rental assistance. With topping up, lump-sum cash transfers are the natural benchmark ("tortilla subsidies" versus Progresa).
- **With positive correlation, subsidies are self-targeting** (e.g. public transport), and topping up leaves the optimum unchanged.
- **Greater dispersion in welfare weights can reduce consumption.** A mean-preserving spread raises consumption under positive correlation, but lowers it under negative correlation with topping up.
- **Technical contribution**: We solve mechanism design problems with FOSD and SOSD constraints caused by type-dependent outside options.

Fin

Appendix

Assumption: no lump-sum cash transfers

When it matters: With topping up, or under positive correlation, the shadow value is $\mu^* = (\mathbf{E}[\omega] - \alpha)_+$. Under negative correlation without topping up, it can be positive even when $\mathbf{E}[\omega] \leq \alpha$.

Possible reasons:

- **Institutional**: subsidies designed by a government agency without tax/transfer powers.
- **Political**: Liscow and Pershing (2022) find US voters prefer in-kind redistribution to cash transfers.
- **Household Economics**: Currie (1994) finds in-kind redistribution has stronger benefits for children than cash transfer programs.
- **Pedagogical**: a positive multiplier measures the gain from relaxing the restriction on cash transfers.
- **Model**: without the NLS constraint, the social planner would want to make unbounded cash transfers when $\mathbf{E}[\omega] > \alpha$.

Back to subsidy design

Topping up $\Longleftarrow$ lower bound (1/2)

Suppose (q, t) satisfies (IC), (IR) and $q(\theta) \geq q^{\text{LF}}(\theta)$. We show that the total subsidy $S(\theta) = cq(\theta) - t(\theta)$ is nonnegative and nondecreasing.

1. $t(\underline{\theta}) \leq cq(\underline{\theta})$ by (IR):

$$t(\underline{\theta}) \leq \underline{\theta}v(q(\underline{\theta})) - \underline{\theta}v(q^{\text{LF}}(\underline{\theta})) + cq^{\text{LF}}(\underline{\theta}),$$

and $\underline{\theta}v(q^{\text{LF}}(\underline{\theta})) - cq^{\text{LF}}(\underline{\theta}) \geq \underline{\theta}v(q(\underline{\theta})) - cq(\underline{\theta})$ by definition of q^{LF}, so $t(\underline{\theta}) \leq cq(\underline{\theta})$.

Next

Back to private market interaction

Topping up $\Longleftarrow$ lower bound (2/2)

 # 2. Payments increase by at most c per additional unit. Choose right-continuous allocations and let $\nu = v \circ q$. By (IC), for $\theta' > \theta$,

$$t(\theta') - t(\theta) = \theta'\nu(\theta') - \theta\nu(\theta) - \int_{\theta}^{\theta'} \nu(s)\,\mathrm{d}s = \int_{(\theta,\theta']} s\,\mathrm{d}\nu(s).$$

On continuous portions, concavity and the lower bound give $sv'(q(s)) \le c$. At a jump at s, $s[v(q(s)) - v(q(s^-))] \le c[q(s) - q(s^-)]$ for the same reason.

Thus $t(\theta') - t(\theta) \le c[q(\theta') - q(\theta)]$: the total subsidy is nondecreasing. Together with step 1, this permits implementation with topping up.

▸ Previous

◂ Back to private market interaction

Theorem 1: a small subsidy

Let $\varepsilon > 0$ be a small subsidy, and let ΔW denote its welfare effect.

With topping up: target the laissez-faire quantity of type $\hat{\theta}$. Every higher type can claim the subsidy and supplement privately:

$$\Delta W = \varepsilon\,[1 - F(\hat{\theta})]\big(\mathbf{E}[\omega(\theta) \mid \theta \geq \hat{\theta}] - \alpha\big) + O(\varepsilon^{3/2}).$$

Without topping up: choose an interval I on which $\omega(\theta) > \alpha$, and subsidise its laissez-faire allocations: $\Delta W = \varepsilon \int_I [\omega(\theta) - \alpha]\, \mathrm{d}F(\theta) + O(\varepsilon^{3/2})$.

In each case, nearby types distort consumption to claim the subsidy. Each affected interval has width $O(\sqrt{\varepsilon})$, and its total welfare effect is $O(\varepsilon^{3/2})$.

A positive first-order redistributive gain dominates these local distortions.

Necessity

Back: without topping up

Back: with topping up

Theorem 1: why the conditions are necessary

Write $\Delta U(\theta) = U(\theta) - U^{\mathrm{LF}}(\theta)$ and $\Delta v(\theta) = v(q(\theta)) - v(q^{\mathrm{LF}}(\theta))$. Let $T(\theta) = \int_{\theta}^{\overline{\theta}} [\omega(s) - \alpha]\, \mathrm{d}F(s)$.

With topping up, the envelope theorem gives

$$\Delta W = T(\underline{\theta})\Delta U(\underline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} T(\theta)\Delta v(\theta)\, \mathrm{d}\theta$$
$$+ \alpha \int_{\underline{\theta}}^{\overline{\theta}} \left[\theta v(q(\theta)) - cq(\theta) - U^{\mathrm{LF}}(\theta)\right] \mathrm{d}F(\theta).$$

If $T \leq 0$, all three terms are nonpositive: (IR) gives $\Delta U(\underline{\theta}) \geq 0$, (TU) gives $\Delta v \geq 0$, and laissez-faire maximises total surplus pointwise.

Without topping up, type θ's contribution to ΔW is

$$\alpha\left[\theta v(q(\theta)) - cq(\theta) - U^{\mathrm{LF}}(\theta)\right] + [\omega(\theta) - \alpha]\Delta U(\theta).$$

If $\omega \leq \alpha$, both terms are nonpositive by efficiency and (IR).

▸ Small-subsidy argument ◂ Back: without topping up ◂ Back: with topping up

Theorem 2: the exact coincidence conditions

Let $\mu^* = (\mathbf{E}[\omega] - \alpha)_+$ and define

$$\theta^L = \min\left\{\hat\theta \in [\underline{\theta}, \overline{\theta}] : \int_{\hat\theta}^{\overline{\theta}} [\omega(s) - \alpha]\, dF(s) \geq 0\right\}.$$

The generalised virtual valuation is $J_\mu(\theta) = J(\theta) + \mu\underline{\theta}\,\delta_{\underline{\theta}}(\theta)/[\alpha f(\theta)]$, where $\delta_{\underline{\theta}}$ is a point mass at the lowest type.

Theorem 2. The optimal mechanisms coincide if and only if

$$\begin{cases} \omega(\theta) \leq \alpha, & \theta \in [\underline{\theta}, \theta^L), \\ \overline{J_{\mu^*}|_{[\theta^L,\overline{\theta}]}}(\theta) \geq \theta, & \theta \in [\theta^L, \overline{\theta}]. \end{cases}$$

The bar denotes ironing over the indicated interval.

Construction. Leave types below θ^L at laissez-faire; at and above it, use $D(c, \overline{J_{\mu^*}|_{[\theta^L,\overline{\theta}]}}(\theta))$.

Back to coincidence

Characterising the optimal subsidy with topping up

Suppose ω is decreasing. If $\alpha \geq \mathbf{E}[\omega]$, the optimum is laissez-faire. For $\alpha < \mathbf{E}[\omega]$, write $\mu^*_{\mathsf{TU}} = (\mathbf{E}[\omega] - \alpha)_+$ and $\widetilde{J} = J_{\mu^*_{\mathsf{TU}}}$, including the bottom atom.

Theorem. The optimal allocation rule is unique, continuous and satisfies

$$q^*(\theta) = \begin{cases} D\left(c, \overline{\widetilde{J}|_{[\underline{\theta},\theta_\alpha]}}(\theta)\right) & \text{for } \theta \leq \theta_\alpha \\ q^{\mathsf{LF}}(\theta) & \text{for } \theta > \theta_\alpha, \end{cases}$$

where θ_α is defined by

$$\theta_\alpha = \inf\left\{\theta \in \Theta : \overline{\widetilde{J}|_{[\underline{\theta},\theta]}}(\theta) \leq \theta\right\}.$$

If the cutoff set is empty, set $\theta_\alpha = \overline{\theta}$.

Intuition: Below θ_α, consumption weakly exceeds laissez-faire; above it, consumers receive q^{LF}.

▸ Proof ▸ Ironing ◂ Back to allocation comparison

Solving for the optimal mechanism

return to summary

$$\max_{q} \alpha \int_{\underline{\theta}}^{\overline{\theta}} \left[\widetilde{J}(\theta) v(q(\theta)) - cq(\theta)\right] \, \mathrm{d}F(\theta),$$

s.t. q nondecreasing and $q(\theta) \geq q^{\mathrm{LF}}(\theta)$.

Solving for the optimal mechanism

▸ return to summary

$$\max_q \alpha \int_{\underline{\theta}}^{\overline{\theta}} \left[\widetilde{J}(\theta) v(q(\theta)) - cq(\theta) \right] \, \mathrm{d}F(\theta),$$

s.t. q nondecreasing and $q(\theta) \geq q^{\mathrm{LF}}(\theta)$.

Guess 1: Pointwise maximiser

Away from the bottom atom, use $q(\theta) = D(c, J(\theta))$.

Solving for the optimal mechanism

return to summary

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Guess 1: Pointwise maximiser

Away from the bottom atom, use $q(\theta) = D(c, J(\theta))$.

If J is nondecreasing, so is this allocation. If $J(\theta) \geq \theta$, then $q(\theta) \geq q^{\mathrm{LF}}(\theta)$.

The bottom atom must also be included when imposing monotonicity.

Optimality proof

Back to allocation comparison

Solving for the optimal mechanism

return to summary

$$\max_q \alpha \int_{\underline{\theta}}^{\overline{\theta}} \left[\widetilde{J}(\theta) v(q(\theta)) - cq(\theta)\right] \, dF(\theta),$$

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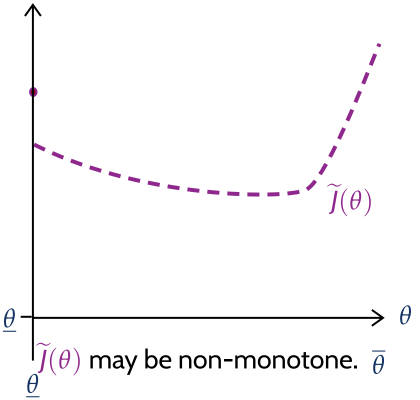

Optimality proof

Back to allocation comparison

Solving for the optimal mechanism

return to summary

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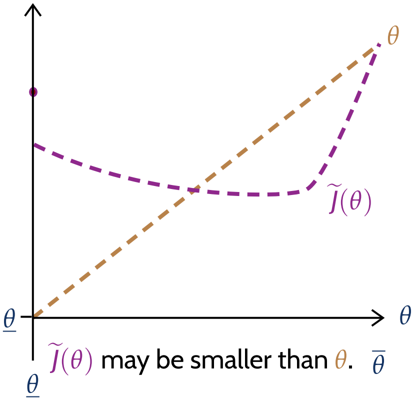


$\widetilde{J}(\theta)$ may be smaller than θ.

Optimality proof

Back to allocation comparison

Solving for the optimal mechanism

return to summary

$$\max_q \alpha \int_{\underline{\theta}}^{\overline{\theta}} \left[\widetilde{J}(\theta)v(q(\theta)) - cq(\theta)\right] \, \mathrm{d}F(\theta),$$

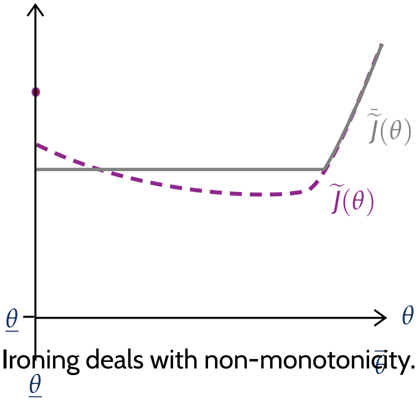
s.t. q nondecreasing and $q(\theta) \geq q^{\mathrm{LF}}(\theta)$.

Guess 2: Relaxing the (LB) constraint

Toikka (2011); Akbarpour, Dworczak, Kominers (2021)

$$\rightsquigarrow q(\theta) = D(c, \bar{\widetilde{J}}(\theta)),$$

where $\bar{\widetilde{J}}$ pools appropriate intervals at their F-weighted averages, including the bottom atom.



Ironing deals with non-monotonicity.

Ironing

Optimality proof

Back to allocation comparison

Solving for the optimal mechanism

return to summary

$$\max_q \alpha \int_{\underline{\theta}}^{\overline{\theta}} \left[\widetilde{J}(\theta) v(q(\theta)) - c q(\theta)\right] \, \mathrm{d}F(\theta),$$

s.t. q nondecreasing and $q(\theta) \geq q^{\mathrm{LF}}(\theta)$.

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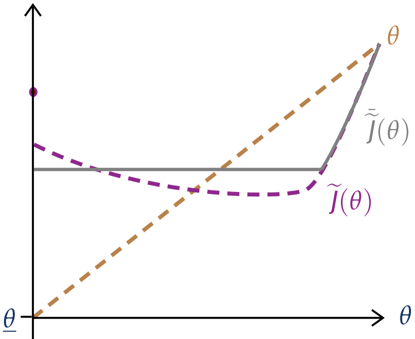


But not the lower-bound constraint $\rightsquigarrow$ interaction.

Ironing

Optimality proof

Back to allocation comparison

Solving for the optimal mechanism

return to summary

$$\max_q \alpha \int_{\underline{\theta}}^{\overline{\theta}} \left[\widetilde{J}(\theta)v(q(\theta)) - cq(\theta)\right] \, \mathrm{d}F(\theta),$$

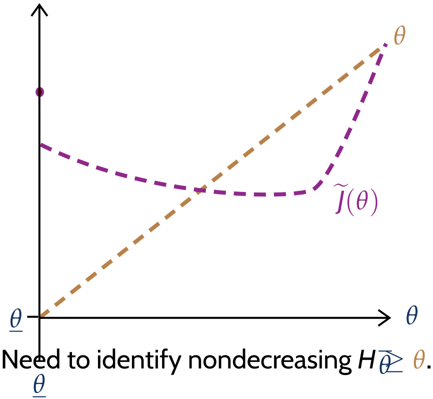
s.t. q nondecreasing and $q(\theta) \geq q^{\mathrm{LF}}(\theta)$.

Guess 3: Our approach

Suppose the solution is of the form

$$q(\theta) = D(c, H(\theta)).$$

Feasibility requires H to be nondecreasing and satisfy $H(\theta) \geq \theta$.



Need to identify nondecreasing $H \geq \theta$.

Ironing

Optimality proof

Back to allocation comparison

Verifying the topping-up construction

Because $q^*(\theta) = D(c, H(\theta))$, for any feasible q

$$\int_\Theta \underbrace{[H(\theta)v(q^*(\theta)) - cq^*(\theta)]}_{\text{surplus of type } H(\theta) \text{ at } D(c,H(\theta))} \, \mathrm{d}F(\theta) \geq \int_\Theta \underbrace{[H(\theta)v(q(\theta)) - cq(\theta)]}_{\text{surplus of type } H(\theta) \text{ at } q(\theta)} \, \mathrm{d}F(\theta).$$

Back to construction
Next
Back to allocation comparison

Verifying the topping-up construction

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We want to show, for any feasible q

$$\underbrace{\int_\Theta [\tilde{J}(\theta)v(q^*(\theta)) - cq^*(\theta)] \, \mathrm{d}F(\theta)}_{\text{objective at } q^*} \geq \underbrace{\int_\Theta [\tilde{J}(\theta)v(q(\theta)) - cq(\theta)] \, \mathrm{d}F(\theta)}_{\text{objective at feasible } q}.$$

◂ Back to construction ▸ Next ◂ Back to allocation comparison

Verifying the topping-up construction

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Subtracting, it suffices to show, for any feasible q

$$\int_\Theta [\tilde{J}(\theta) - H(\theta)][v(q^*(\theta)) - v(q(\theta))] \, \mathrm{d}F(\theta) \geq 0.$$

◂ Back to construction ▸ Next ◂ Back to allocation comparison

Verifying the variational inequality

To show $\int_\Theta [\widetilde{J}(\theta) - H(\theta)][v(q^*(\theta)) - v(q(\theta))] \, \mathrm{d}F(\theta) \geq 0$.

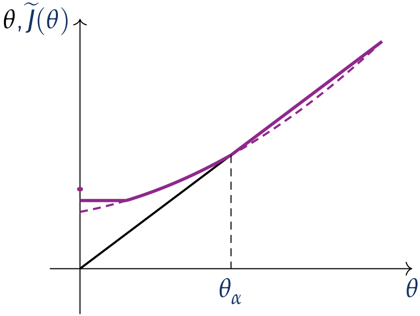
Previous

Back to allocation comparison

Verifying the variational inequality

To show $\int_\Theta [\widetilde{J}(\theta) - H(\theta)][v(q^*(\theta)) - v(q(\theta))] \, \mathrm{d}F(\theta) \geq 0$.

There are three possibilities for H, partitioning Θ into intervals:



Verifying the variational inequality

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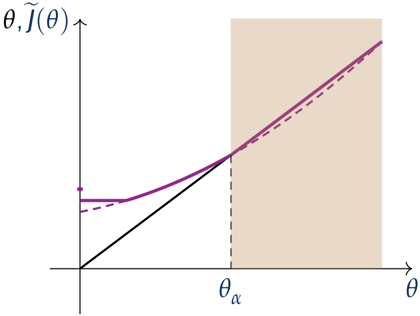
There are three possibilities for H, partitioning Θ into intervals:

1. $H(\theta) = \theta$: by construction $\widetilde{J}(\theta) \leq \theta = H(\theta)$ and $v(q(\theta)) \geq v(q^*(\theta)) \rightsquigarrow$ integrand ≥ 0.



Verifying the variational inequality

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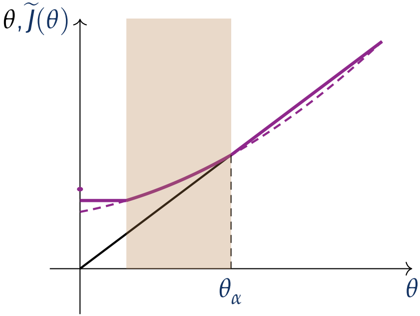


Verifying the variational inequality

To show $\int_\Theta [\widetilde{J}(\theta) - H(\theta)][v(q^*(\theta)) - v(q(\theta))]\, \mathrm{d}F(\theta) \geq 0$.

There are three possibilities for H, partitioning Θ into intervals:

1. $H(\theta) = \theta$: by construction $\widetilde{J}(\theta) \leq \theta = H(\theta)$ and $v(q(\theta)) \geq v(q^*(\theta)) \rightsquigarrow$ integrand ≥ 0.

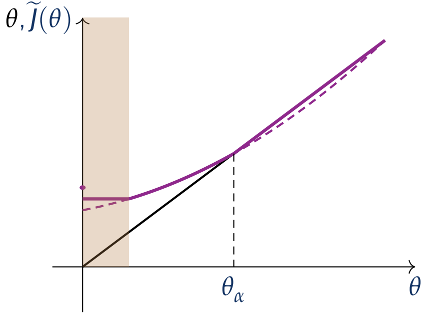
2. $H(\theta) = \widetilde{J}(\theta)$: integrand $= 0$.

3. $H(\theta) = \overline{\widetilde{J}|_{[\underline{\theta},\theta_\alpha]}}(\theta) \neq \widetilde{J}(\theta)$:

technical lemma $\rightsquigarrow$ on any such interval Θ_i, $H = \overline{\widetilde{J}|_{\Theta_i}}$

$\rightsquigarrow$ optimality of $D(c, H(\theta))$ in the problem on Θ_i *without* (LB)

$\Longrightarrow$ the same variational inequality characterises optimality. $\square$



Previous

Back to allocation comparison

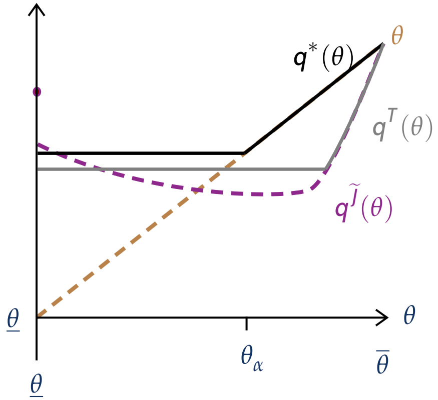
Role of the private market

When we compare the optima with and without the (LB) constraint, $q^*(\theta)$ can exceed $q^T(\theta)$ for *all* types.

$\rightsquigarrow$ Inability to tax can cause upward distortion, even for consumers who would be subsidised in the absence of the (LB) constraint.

It is not optimal to calculate the optimal subsidy/tax schedule and set taxes to zero.

The cutoff and pooling conditions provide global optimality checks beyond pointwise first-order conditions.



◂ Back to allocation comparison

(A) Which consumers go to the private market?

Theorem 4 (NTU cutoff). Suppose ω is decreasing and $\alpha < \max \omega$. Write $m = \mathbf{E}[\omega]$.

- The upper cutoff is
$$\theta^H(\mu^*) := \max \left\{ \theta \in [\underline{\theta}, \overline{\theta}] : \int_{\underline{\theta}}^{\theta} [\omega(s) - \alpha] \, \mathrm{d}F(s) \geq \mu^* \right\}.$$
- If $\mu^* > m - \alpha$, the cutoff is interior and high types choose the private market.
- All types are served when $\mu^* = m - \alpha$; if $m > \alpha$, either case is possible.

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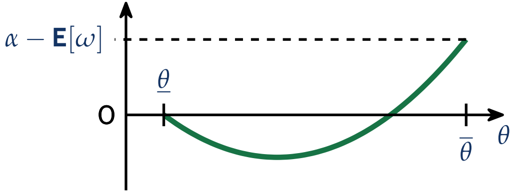
$$\theta \mapsto \int_{\underline{\theta}}^{\theta} [\alpha - \omega(s)] \; dF(s)$$

If $\mathbf{E}[\omega] \leq \alpha$:



Allocation details

Back to allocation comparison

Ⓐ Which consumers go to the private market?

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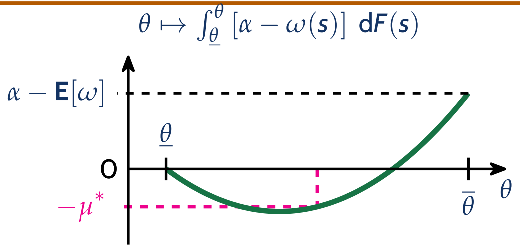

▸ Allocation details

◂ Back to allocation comparison

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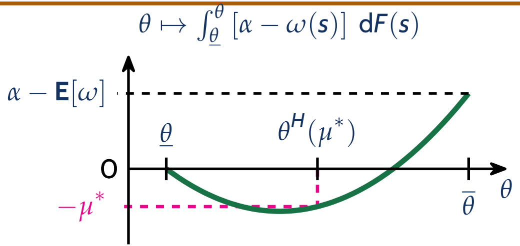


▸ Allocation details

◂ Back to allocation comparison

B Which consumers benefit from in-kind redistribution?

Theorem 4 (NTU allocation). Suppose ω is decreasing and $\alpha < \max \omega$. Write $G(\theta) = \int_{\underline{\theta}}^{\theta} [\omega(s) - \alpha] \, dF(s)$, $m = \mathbf{E}[\omega]$, and $\mu_{\max} = \max_\theta G(\theta)$.

For $\mu \in [(m-\alpha)_+, \mu_{\max}]$, let $\theta^H(\mu) = \max\{\theta : G(\theta) \geq \mu\}$ and define

$$H_\mu(\theta) = J_\mu(\theta) + \frac{\mu + \alpha - m}{\alpha f(\theta)}, \qquad q_\mu(\theta) = D\left(c, \overline{H_\mu|_{[\underline{\theta}, \theta^H(\mu)]}}(\theta)\right).$$

Here $J_\mu = J + \mu \underline{\theta} \delta_{\underline{\theta}} / (\alpha f)$ includes the bottom atom; ironing is restricted to the served interval.

Let $U^H(\mu) = \underline{\theta} v(q_\mu(\underline{\theta})) + \int_{\underline{\theta}}^{\theta^H(\mu)} v(q_\mu(s)) \, ds$. Then

$$\mu^* = \min\left\{\mu \in [(m-\alpha)_+, \mu_{\max}] : U^H(\mu) \geq U^{LF}(\theta^H(\mu))\right\}.$$

The optimum is $q^* = q_{\mu^*}$ up to $\theta^H(\mu^*)$ and $q^* = q^{LF}$ above it. A free public option is used if and only if $\mu^* > 0$.

▸ Cutoff details

◂ Back to allocation comparison

Private market taxation

Our baseline model assumes the planner cannot tax the private market.

Taxation of the private market reduces consumers' outside option, relaxing the (LB) constraint. If taxation is costly (*e.g.*, because of distortions affecting ineligible consumers):

Proposition. Suppose the planner faces a convex cost $\Gamma(\tau)$ for taxation of the private market. Then there exist an optimal tax level τ^* and a subsidy program for eligible consumers satisfying

$$q^*(\theta) = D(c, H_{\tau^*}(\theta)),$$

where $H_{\tau^*}(\theta) \leq H(\theta)$.

Back to subsidy design

Budget constraints and endogenous welfare weights

In our baseline model, $\omega(\cdot)$ and α are taken exogenously.

Our model can be extended to allow weights to be endogenous (cf. Pai and Strack, 2024):

- $\alpha \Longleftrightarrow$ Lagrange multiplier on the social planner's budget constraint.
- $\omega(\theta) \Longleftrightarrow$ the marginal value of money for a consumer with concave preferences

$$\varphi\left(\theta v(q) + I - t\right),$$

and income $I \sim G_\theta$, known but not observed by the social planner, giving

$$\omega(\theta) = \mathbf{E}_{I \sim G_\theta}\left[\varphi'\left(\theta v(q(\theta)) + I - t(\theta)\right)\right].$$

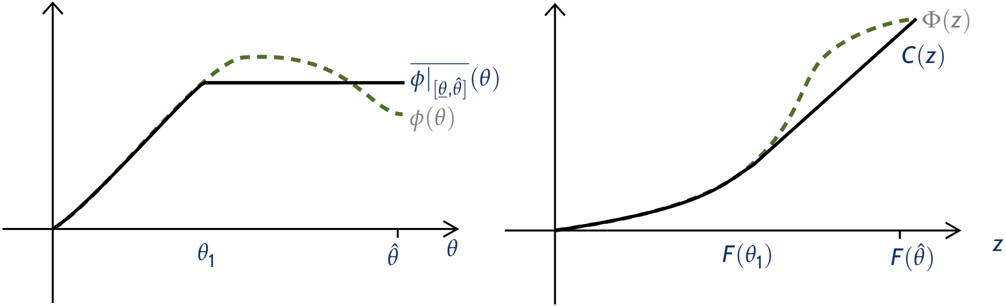
Back to redistributive objective

Ironing

Iron ϕ on $[\underline{\theta}, \hat{\theta}]$ using the quantile $z = F(\theta)$. Set $\Phi(0) = 0$ and, for $z > 0$, let $\Phi(z) = \int_{[\underline{\theta}, F^{-1}(z)]} \phi(s)\, dF(s)$, including any bottom atom.

Let C be the greatest convex minorant of Φ on $[0, F(\hat{\theta})]$. Then $\overline{\phi|_{[\underline{\theta},\hat{\theta}]}}(\theta) = C'_+(F(\theta))$ at interior types, with endpoint values given by limits.

Linear segments of C pool the corresponding types at their F-weighted average virtual valuation.



Proof

Statement

Back to positive proof

Back to allocation comparison

Allocation comparison: a solved example

Let $\theta \sim U[0.1, 10.1]$, $v(q) = \log q$, and $c = 1$.

Welfare weights are $\omega(\theta) = 0.03 + 3e^{-0.4(\theta-0.1)}$, with $\alpha = 0.57 < \mathbf{E}[\omega] \simeq 0.7663$.

	Topping up	No topping up
(LS) multiplier μ	0.1963	0.2625
Free quantity	1.7946	2.1187
End of pooling	1.7946	4.2369
Upper cutoff	1.7946	8.6783

With topping up, $q^*_{\mathrm{TU}}(\theta) = \max\{1.7946, \theta\}$.

Without topping up, $q^*(\theta) = 2.1187$ until $\theta = 4.2369$, then follows the increasing, convex virtual type until $\theta = 8.6783$.

The allocations cross at $\theta \simeq 2.1187$, during NTU pooling.

The dot marks the additional (LS) atom at $\underline{\theta}$. Its mass in $J_\mu\, dF$ is $\mu\underline{\theta}/\alpha$: 0.03443 with TU and 0.04606 with NTU.

Back to allocation comparison

Negative correlation: the higher-α example

Let $\theta \sim U[0.1, 10.1]$, $v(q) = \log q$, $c = 1$, and $\omega(\theta) = 0.03 + 3e^{-0.4(\theta - 0.1)}$.

Here $\alpha = 0.85 > \mathbf{E}[\omega] \simeq 0.7663$. With topping up, $q^*_{\mathrm{TU}} = q^{\mathrm{LF}}$.

No topping up	
(LS) multiplier μ	0.1729
Constant shift $K/(\alpha f)$	3.0193
Free quantity	1.4249
End of pooling	2.5329
Upper cutoff θ^H	6.4026

From the end of pooling to θ^H, $q^*(\theta) = J(\theta) + 3.0193$. At θ^H, the allocation reaches laissez-faire and (IR) binds.

The bottom atom has mass $\mu\underline{\theta}/\alpha \simeq 0.02034$ in $J_\mu \, dF$. The next main slide uses the same primitives with $\alpha = 0.57$.

Back to intervention comparison

Positive correlation: numerical example

Let $\theta \sim U[1, 11]$, $v(q) = \log q$, $c = 1$, and $\omega(\theta) = 0.1 + 0.15\theta$. Then $\mathbf{E}[\omega] = 1$ and $J(\theta) = \theta + \left[(0.1 - \alpha)(11 - \theta) + 0.075(121 - \theta^2)\right] / \alpha$.

	$\alpha = 1.15$	$\alpha = 0.9$
(LS) multiplier μ	0	0.1
Lowest subsidised type θ^L	3	1
Free quantity	—	4.0302
End of free pooling θ_b	—	2.1819

In both cases, J is increasing and the TU and NTU mechanisms coincide.

For $\alpha = 0.9$, the atom has mass $\mu\underline{\theta}/\alpha = 1/9$ in $J_\mu \, \mathrm{d}F$. Its ironing determines θ_b through $\int_1^{\theta_b} [J(\theta_b) - J(s)] \, \mathrm{d}s = 10\mu/\alpha$.

Back: $\alpha = 1.15$

Back: $\alpha = 0.9$

Proposition 3: comparing the two mechanisms

Suppose ω is decreasing. Let q^*_{TU} and q^* denote optimal total consumption with and without topping up.

Multipliers and cutoffs: $\mu^*_{\mathrm{TU}} \leq \mu^*$ and $\theta^H_{\mathrm{TU}}(\mu^*_{\mathrm{TU}}) \leq \theta^H(\mu^*)$.

Allocation: there exists $\hat{\theta} \in [\underline{\theta}, \overline{\theta}]$ such that

$$\begin{cases} q^*(\theta) \geq q^*_{\mathrm{TU}}(\theta), & \theta < \hat{\theta}, \\ q^*(\theta) \leq q^*_{\mathrm{TU}}(\theta), & \theta > \hat{\theta}. \end{cases}$$

All comparisons are weak and allow equality throughout.

The upper cutoffs concern total consumption. Above the topping-up cutoff, consumers may retain an inframarginal subsidy while supplementing privately.

Back to economic implications

Comparative statics: higher welfare weights

Propositions 1 and 4. Let $\omega_H \geq \omega_L$ pointwise, with both functions increasing or both decreasing as indicated below. Fix F, v, c, α; write $\Delta x = x_H - x_L$.

Correlation	Mechanism	Multiplier	Cutoff	Total consumption
Positive	Both	$\Delta\mu^* \geq 0$	$\Delta\theta^L \leq 0$	$\Delta q^*(\theta) \geq 0$
Negative	Topping up	$\Delta\mu^*_{\mathrm{TU}} \geq 0$	$\Delta\theta^H_{\mathrm{TU}} \geq 0$	$\Delta q^*_{\mathrm{TU}}(\theta) \geq 0$
Negative	No topping up	$\Delta\mu^* \geq 0$	$\Delta\theta^H \geq 0$	Single-crossing $(*)$

$(*)$ There exists $\hat{\theta} \in [\underline{\theta}, \overline{\theta}]$ such that $\Delta q^*(\theta) \geq 0$ for $\theta < \hat{\theta}$, and $\Delta q^*(\theta) \leq 0$ for $\theta > \hat{\theta}$.

All consumption comparisons hold at every indicated type. Upper cutoffs are evaluated at their corresponding optimal multipliers.

Back: positive correlation   Virtual types   Back: negative correlation

Comparative statics: mean-preserving spreads

Propositions 2 and 5. Let $\mathbf{E}[\omega_H] = \mathbf{E}[\omega_L]$ and, for every θ,
$\int_{\underline{\theta}}^{\overline{\theta}} [\omega_H(s) - \omega_L(s)]\, dF(s) \geq 0$ if both weights increase in θ, and ≤ 0 if both decrease.
Fix F, v, c, α; write $\Delta x = x_H - x_L$.

Correlation	Mechanism	Multiplier	Cutoff	Total consumption
Positive	Both	$\Delta\mu^* = 0$	$\Delta\theta^L \leq 0$	$\Delta q^*(\theta) \geq 0$
Negative	Topping up	$\Delta\mu^*_{\mathrm{TU}} = 0$	$\Delta\theta^H_{\mathrm{TU}} \leq 0$	$\Delta q^*_{\mathrm{TU}}(\theta) \leq 0$
Negative	No topping up	$\Delta\mu^* \geq 0$	Ambiguous	Ambiguous

Consumption inequalities hold pointwise; upper cutoffs use their own optimal multipliers.

With topping up, dispersion strengthens self-targeting under positive correlation and leakage under negative correlation.

Back: positive correlation Virtual types Back: negative correlation

Comparative statics: diagrammatic intuition

Hold F, v, c, α fixed. Subscripts L and H denote the original and changed welfare weights.

The regular virtual valuation depends on the welfare weight in the upper tail:

$$J(\theta) = \theta + \frac{1}{\alpha f(\theta)} \int_{\theta}^{\overline{\theta}} [\omega(s) - \alpha] \, dF(s).$$

The bottom atom

The no-cash constraint adds mass $\mu^* \underline{\theta}/\alpha$ at $\underline{\theta}$ to the measure $J \, dF$.

Ironing spreads this mass over the initial pool.

Participation without topping up

Under negative correlation, the regular coefficient is $h(\theta) = J(\theta) + K/[\alpha f(\theta)]$ on the served region, where $K = \mu^* + \alpha - \mathbf{E}[\omega] \geq 0$.

Higher weights: positive

Higher weights: negative

Spread: positive

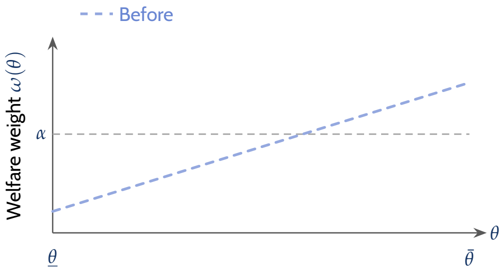
Spread: negative

Higher welfare weights raise every upper-tail integral

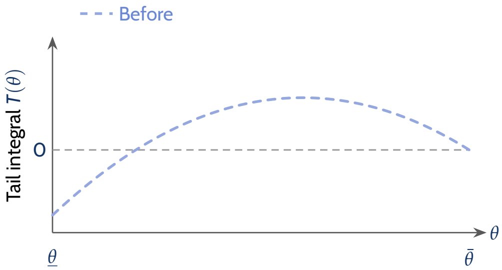
Positive correlation: both mechanisms

Let increasing ω_H, ω_L satisfy $\omega_H \geq \omega_L$ pointwise.

Welfare weights



Cumulative welfare surplus



More weight on every consumer raises the value of subsidising each upper tail.

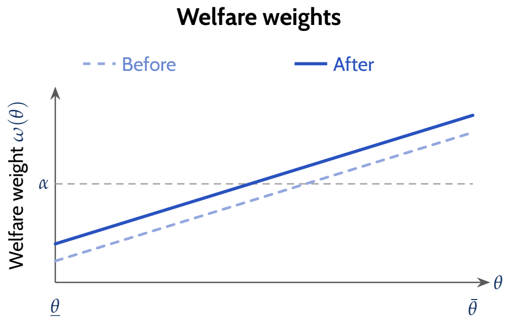
Write $T_i(\theta) = \int_\theta^{\bar{\theta}} [\omega_i(s) - \alpha]\, dF(s)$. Then $T_H \geq T_L$.

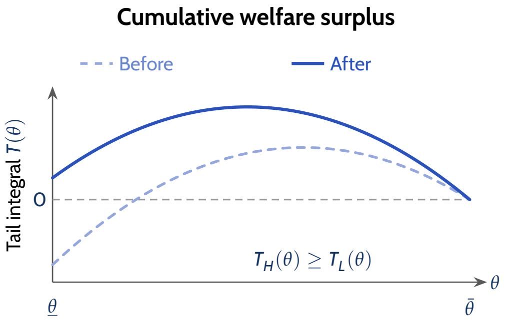
Back to result
Virtual valuations
Continue
Details

Higher welfare weights raise every upper-tail integral

Positive correlation: both mechanisms

Let increasing ω_H, ω_L satisfy $\omega_H \geq \omega_L$ pointwise.





More weight on every consumer raises the value of subsidising each upper tail.

Write $T_i(\theta) = \int_\theta^{\bar{\theta}} [\omega_i(s) - \alpha]\, dF(s)$. Then $T_H \geq T_L$.

The lower service cutoff moves left: upper-tail subsidies become worthwhile for more types.

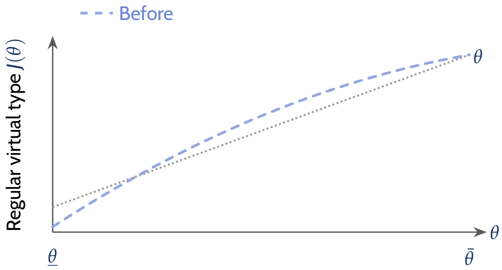
Back to result
Virtual valuations
Continue
Details

Higher virtual valuations raise consumption

Positive correlation: both mechanisms

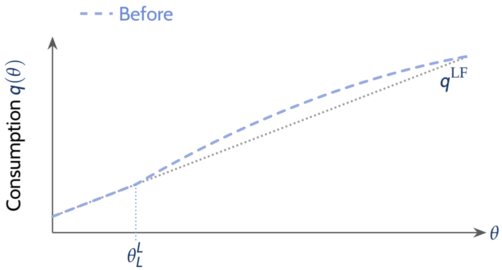
$\Delta J(\theta) = \Delta T(\theta) / [\alpha f(\theta)] \geq 0$ and $\mu_H^* \geq \mu_L^*$.

Virtual valuations



The regular curve rises. If the mean crosses α, a positive bottom atom appears.

Optimal consumption



Ironing incorporates the new atom into a free public option in this example.

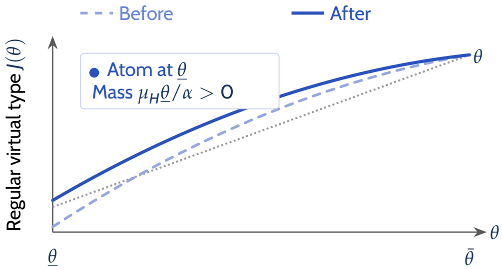
Back to result
Virtual valuations
Details

Higher virtual valuations raise consumption

Positive correlation: both mechanisms

$\Delta J(\theta) = \Delta T(\theta)/[\alpha f(\theta)] \geq 0$ and $\mu_H^* \geq \mu_L^*$.

Virtual valuations



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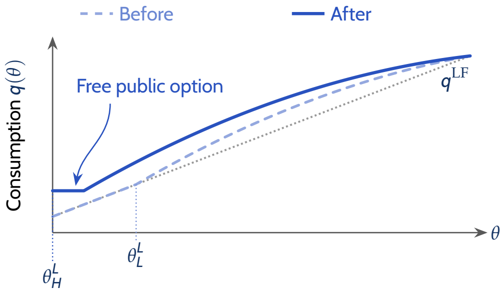
Optimal consumption



Ironing incorporates the new atom into a free public option in this example.

Proposition 1. Consumption weakly rises at every type, $\theta_H^L \leq \theta_L^L$, and $\mu_H^* \geq \mu_L^*$.

Back to result

Virtual valuations

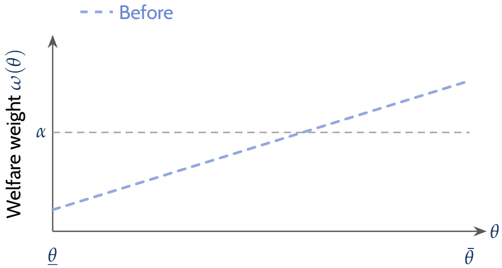
Details

A spread towards high types raises upper-tail weights

Positive correlation: both mechanisms

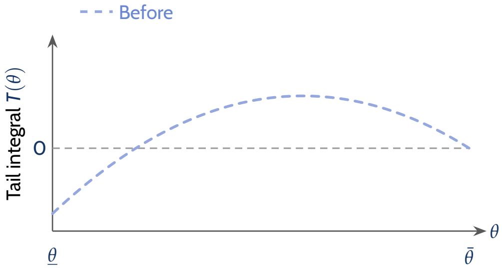
Keep $\mathbf{E}[\omega_H] = \mathbf{E}[\omega_L]$ and require $\int_{\theta}^{\bar{\theta}} (\omega_H - \omega_L)\, dF \geq 0$ for every θ.

Welfare weights



Welfare weight shifts towards consumers with higher demand.

Cumulative welfare surplus



The mean stays fixed, while every upper-tail surplus weakly rises.

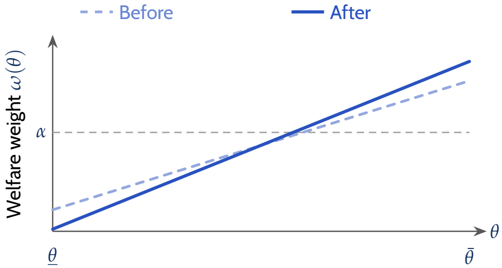
◀ Back to result ▶ Virtual valuations ▶ Continue ▶ Details

A spread towards high types raises upper-tail weights

Positive correlation: both mechanisms

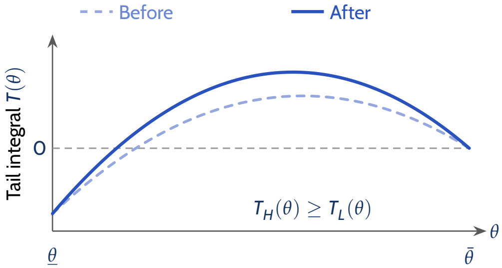
Keep $\mathbf{E}[\omega_H] = \mathbf{E}[\omega_L]$ and require $\int_\theta^{\bar{\theta}} (\omega_H - \omega_L)\, dF \geq 0$ for every θ.

Welfare weights



Welfare weight shifts towards consumers with higher demand.

Cumulative welfare surplus



The mean stays fixed, while every upper-tail surplus weakly rises.

A subsidy that attracts higher types now reaches consumers with greater redistributive priority.

Back to result
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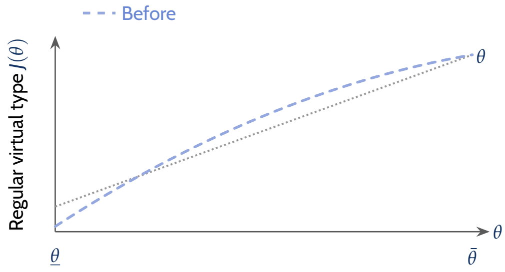
A spread raises consumption with the same bottom atom

Positive correlation: both mechanisms

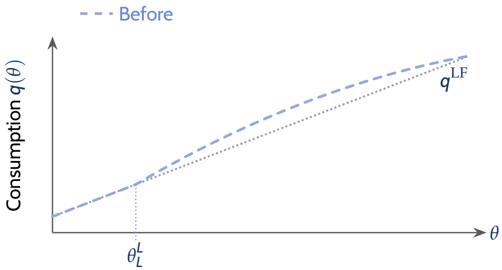
$\Delta J(\theta) \geq 0$ and $\Delta\mu^* = 0$, since the mean welfare weight is unchanged.

Virtual valuations



The regular virtual valuation rises. The bottom atom has the same mass in both cases.

Optimal consumption



The programme expands towards lower types, and consumption weakly rises.

◂ Back to result

▸ Virtual valuations

▸ Details

A spread raises consumption with the same bottom atom

Positive correlation: both mechanisms

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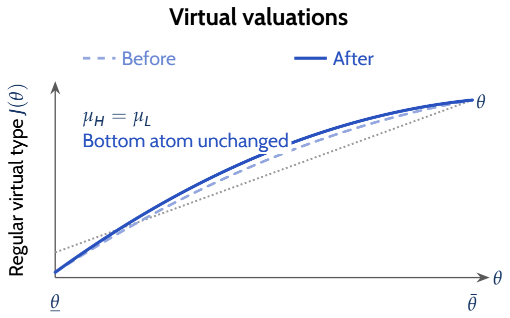
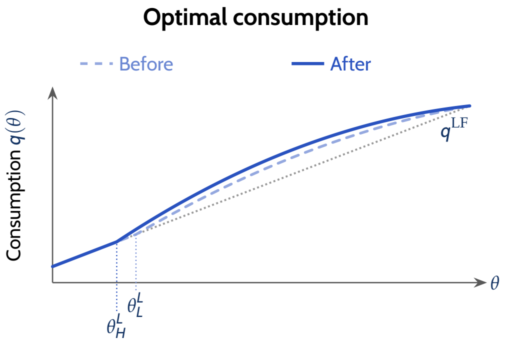


The regular virtual valuation rises. The bottom atom has the same mass in both cases.



The programme expands towards lower types, and consumption weakly rises.

Proposition 2. $q_H^*(\theta) \geq q_L^*(\theta)$ at every type, $\theta_H^L \leq \theta_L^L$, and $\mu_H^* = \mu_L^*$.

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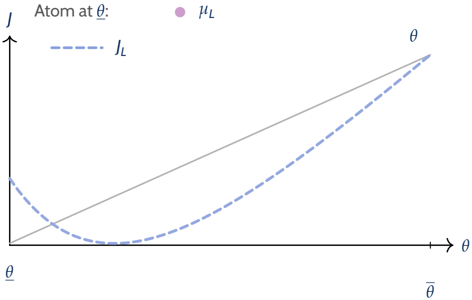
Details

Higher welfare weights expand provision with topping up

Negative correlation: higher weights

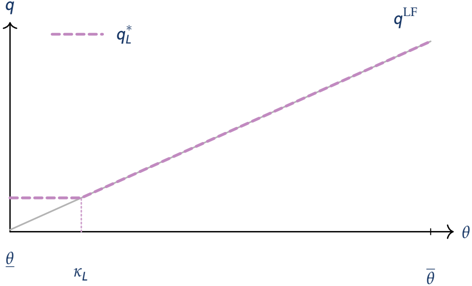
Let decreasing ω_H, ω_L satisfy $\omega_H \geq \omega_L$. Both J and the bottom atom weakly rise.

Virtual valuations



More weight on every upper tail makes leakage to higher types less costly.

Optimal consumption



The upper service cutoff κ moves right. Consumption rises subject to $q \geq q^{\mathrm{LF}}$.

◂ Back to result ▸ Virtual valuations ▸ Continue ▸ Details

Higher welfare weights expand provision with topping up

Negative correlation: higher weights

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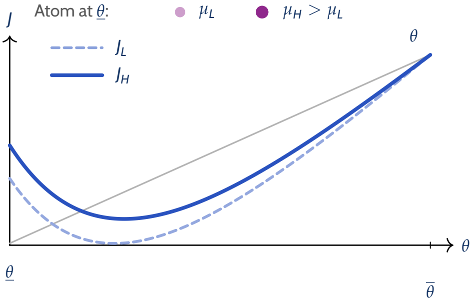
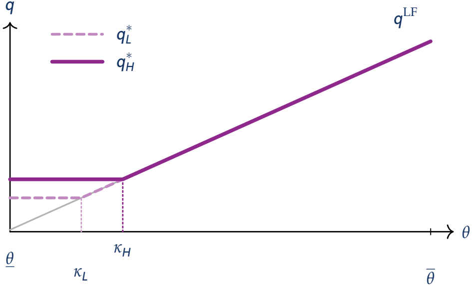
Virtual valuations



More weight on every upper tail makes leakage to higher types less costly.

Optimal consumption



The upper service cutoff κ moves right. Consumption rises subject to $q \geq q^{\mathrm{LF}}$.

Proposition 4, TU. $q^*_{\mathrm{TU},H} \geq q^*_{\mathrm{TU},L}$ pointwise, $\kappa_H \geq \kappa_L$, and $\mu^*_{\mathrm{TU},H} \geq \mu^*_{\mathrm{TU},L}$.

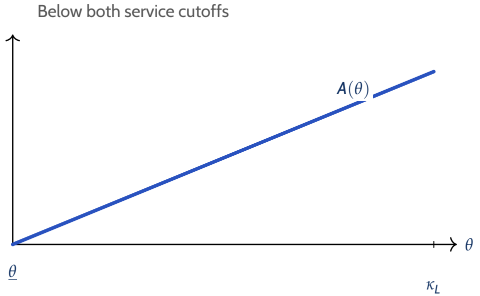
◂ Back to result ▸ Virtual valuations ▸ Continue ▸ Details

The bottom multiplier competes with cumulative welfare weights

Negative correlation: higher weights without topping up

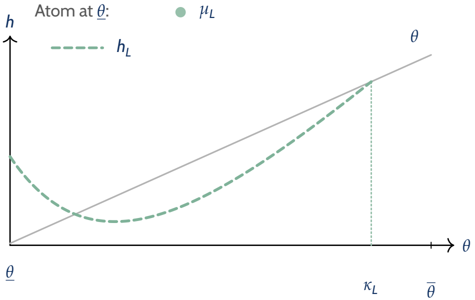
Below both cutoffs, away from the atom, $\alpha f(\theta)\Delta h(\theta) = \Delta\mu^* - A(\theta)$, where $A(\theta) = \int_{\underline{\theta}}^{\theta}(\omega_H - \omega_L)\, dF$.

The sign of the change



The constant $\Delta\mu^*$ competes with an increasing $A(\theta)$. Their difference changes sign at most once.

Effective virtual valuations



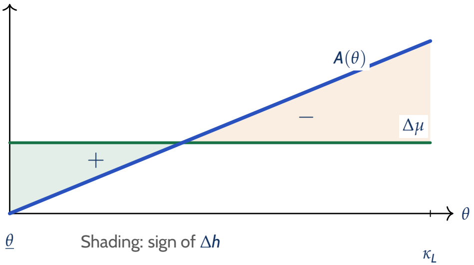
The regular valuation can rise and then fall. The larger atom also enters the ironing calculation.

The bottom multiplier competes with cumulative welfare weights

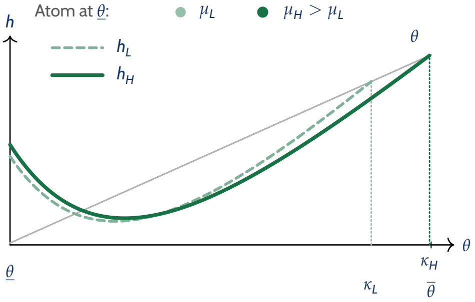
Negative correlation: higher weights without topping up

Below both cutoffs, away from the atom, $\alpha f(\theta)\Delta h(\theta) = \Delta\mu^* - A(\theta)$, where $A(\theta) = \int_{\underline{\theta}}^{\theta}(\omega_H - \omega_L)\,\mathrm{d}F$.

The sign of the change



Effective virtual valuations



The constant $\Delta\mu^*$ competes with an increasing $A(\theta)$. Their difference changes sign at most once.

The regular valuation can rise and then fall. The larger atom also enters the ironing calculation.

In this example, the multiplier effect dominates near the bottom. Further up, the cumulative welfare-weight increase dominates.

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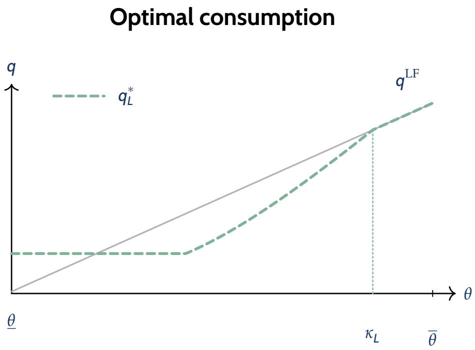
Details

Consumption can rotate towards low types

Negative correlation: higher weights without topping up



The planner raises low types' utility and can use lower quantities further up to screen.

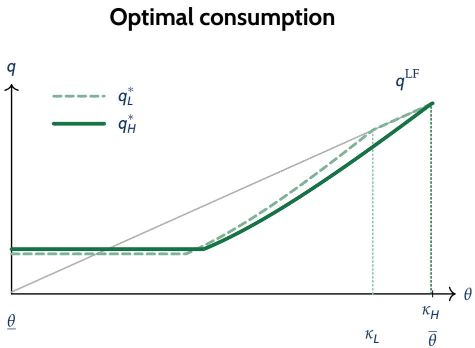
Since $U'(\theta) = v(q(\theta))$, lower consumption reduces the extra utility higher types receive relative to low types.

The theorem allows one crossing or zero. Beyond both service cutoffs, both allocations equal laissez-faire.

◂ Back to result ▸ Virtual valuations ▸ Details

Consumption can rotate towards low types

Negative correlation: higher weights without topping up



The planner raises low types' utility and can use lower quantities further up to screen.

Since $U'(\theta) = v(q(\theta))$, lower consumption reduces the extra utility higher types receive relative to low types.

The bottom multiplier rises, while the positive participation correction K falls. The direct rise in J also matters.

Proposition 4, NTU. $\mu_H^* \geq \mu_L^*$ and $\kappa_H \geq \kappa_L$. For some $\hat{\theta}$, $q_H^* \geq q_L^*$ below $\hat{\theta}$ and $q_H^* \leq q_L^*$ above it.

The theorem allows one crossing or zero. Beyond both service cutoffs, both allocations equal laissez-faire.

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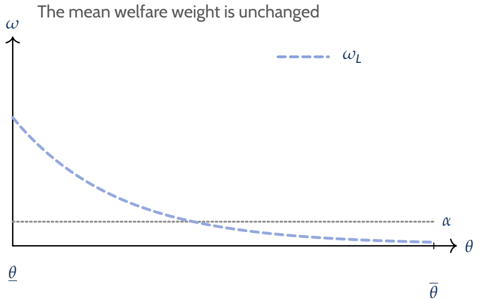
A spread towards low types lowers upper-tail weights

Negative correlation: a mean-preserving spread

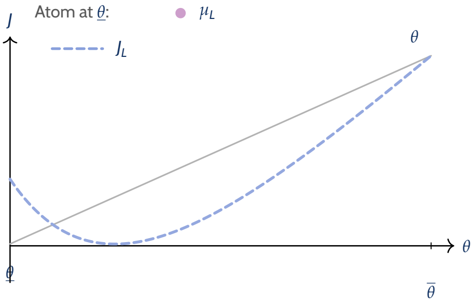
Keep the mean fixed and require $A(\theta) = \int_{\underline{\theta}}^{\theta} (\omega_H - \omega_L)\, dF \geq 0$ for every θ.

Welfare weights



Welfare weight shifts towards consumers with lower demand.

Virtual valuations with topping up



Equal means give $\Delta J(\theta) = -A(\theta)/[\alpha f(\theta)] \leq 0$. The bottom atom stays fixed.

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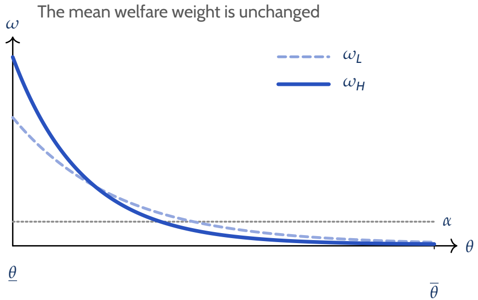
A spread towards low types lowers upper-tail weights

Negative correlation: a mean-preserving spread

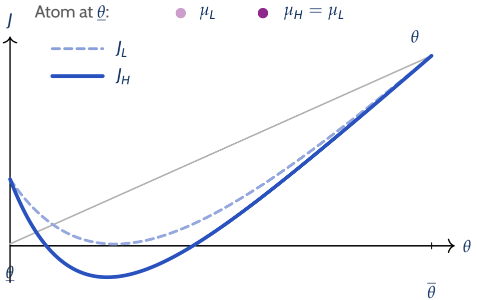
Keep the mean fixed and require $A(\theta) = \int_{\underline{\theta}}^{\theta} (\omega_H - \omega_L)\, \mathrm{d}F \geq 0$ for every θ.

Welfare weights



Welfare weight shifts towards consumers with lower demand.

Virtual valuations with topping up



Equal means give $\Delta J(\theta) = -A(\theta)/[\alpha f(\theta)] \leq 0$. The bottom atom stays fixed.

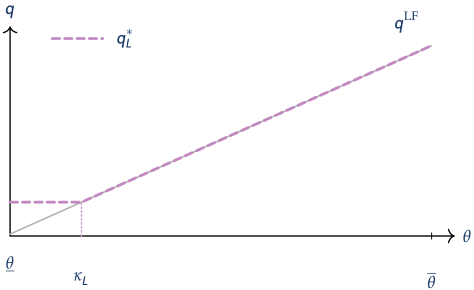
Subsidies to low types still benefit higher types who top up. Those higher types now have lower relative welfare weights.

◂ Back to result ▸ Virtual valuations ▸ Continue ▸ Details

A spread reduces provision with topping up

Negative correlation: a mean-preserving spread

Optimal consumption



The regular virtual valuation falls with the bottom atom held fixed.

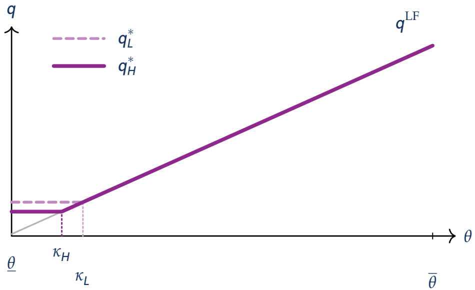
Ironing and the constraint $q \geq q^{\text{LF}}$ translate this into weakly lower consumption.

◂ Back to result ▸ Virtual valuations ▸ Continue ▸ Details

A spread reduces provision with topping up

Negative correlation: a mean-preserving spread

Optimal consumption



The regular virtual valuation falls with the bottom atom held fixed.

Ironing and the constraint $q \geq q^{\text{LF}}$ translate this into weakly lower consumption.

The upper service cutoff moves left. In this example, the free public option provides a smaller quantity.

Proposition 5, TU. $q^*_{\text{TU},H}(\theta) \leq q^*_{\text{TU},L}(\theta)$ at every type, $\kappa_H \leq \kappa_L$, and $\mu^*_{\text{TU},H} = \mu^*_{\text{TU},L}$.

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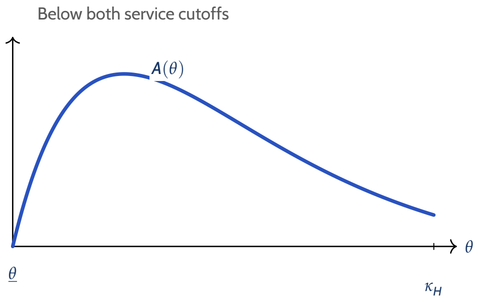
A spread creates competing changes without topping up

Negative correlation: a mean-preserving spread

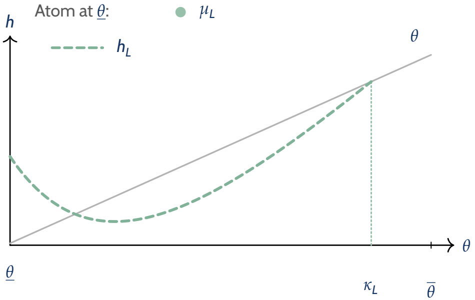
Again $\alpha f(\theta)\Delta h(\theta) = \Delta\mu^* - A(\theta)$ below both cutoffs, away from the atom.

The sign of the change



Here $A \geq 0$ returns to zero at the top of the full type support. Its shape permits multiple sign changes.

Effective virtual valuations



The bottom atom grows, while regular valuations move in both directions in this example.

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Virtual valuations

Continue

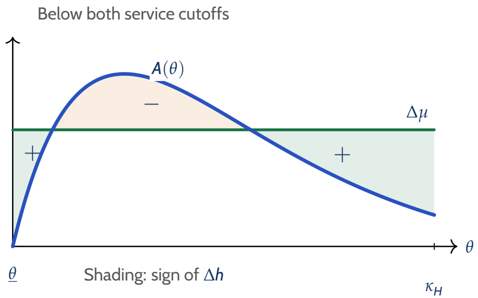
Details

A spread creates competing changes without topping up

Negative correlation: a mean-preserving spread

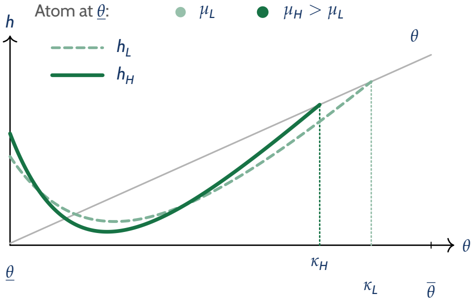
Again $\alpha f(\theta)\Delta h(\theta) = \Delta\mu^* - A(\theta)$ below both cutoffs, away from the atom.

The sign of the change



Here $A \geq 0$ returns to zero at the top of the full type support. Its shape permits multiple sign changes.

Effective virtual valuations



The bottom atom grows, while regular valuations move in both directions in this example.

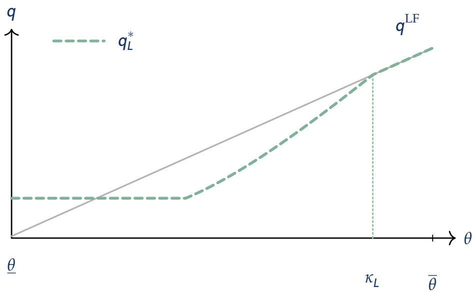
The increase in the bottom multiplier competes with the redistribution of welfare weight towards low types.

◂ Back to result
▸ Virtual valuations
▸ Continue
▸ Details

A larger atom can accompany a smaller free quantity

Negative correlation: a mean-preserving spread without topping up

Optimal consumption in the example



Ironing averages the larger atom together with the changed regular valuations.

Proposition 5, NTU. $\mu_H^* \geq \mu_L^*$. Consumption and the upper service cutoff have ambiguous responses.

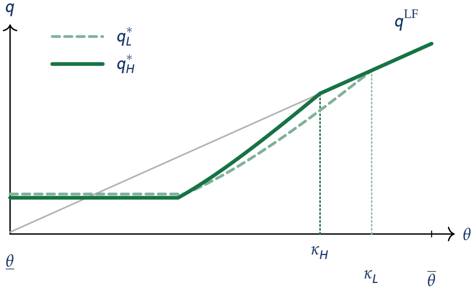
The multiplier measures the value of relaxing the no-cash constraint. The free quantity depends on the entire ironing calculation.

◂ Back to result ▸ Virtual valuations ▸ Details

A larger atom can accompany a smaller free quantity

Negative correlation: a mean-preserving spread without topping up

Optimal consumption in the example



Ironing averages the larger atom together with the changed regular valuations.

In this example, the free quantity and the service cutoff fall, even though μ^* rises.

Proposition 5, NTU. $\mu_H^* \geq \mu_L^*$. Consumption and the upper service cutoff have ambiguous responses.

The multiplier measures the value of relaxing the no-cash constraint. The free quantity depends on the entire ironing calculation.

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Positive correlation: identities and calibration

Both mechanisms coincide. Define $T_i(\theta) = \int_\theta^{\overline{\theta}} [\omega_i(s) - \alpha] \, dF(s)$. Then $J_i = \theta + T_i/(\alpha f)$ and $\mu_i^* = (\mathbf{E}[\omega_i] - \alpha)_+$.

The lower cutoff is $\theta_i^L = \min\{\theta : T_i(\theta) \geq 0\}$. Iron the generalised virtual valuation over $[\theta_i^L, \overline{\theta}]$ and use laissez-faire below it.

Numerical figures. $\theta \sim U[1, 11]$, $v(q) = \log q$, $c = 1$, $\alpha = 1.15$.

	Baseline	Higher weights	Spread
$\omega(\theta)$	$0.1 + 0.15\theta$	$0.3 + 0.15\theta$	$1 + 0.195(\theta - 6)$
$\mathbf{E}[\omega]$	1	1.2	1
μ^*	0	0.05	0
θ^L	3	1	2.5385
Free quantity	—	2.6055	—
End of free pooling	—	1.7506	—

Back: higher weights
Main: higher weights
Back: spread
Main: spread

Negative correlation: the two multiplier channels

Write $m_i = \mathbf{E}[\omega_i]$, $G_i(\theta) = \int_{\underline{\theta}}^{\theta} [\omega_i(s) - \alpha]\, \mathrm{d}F(s)$, and let κ_i denote the relevant upper service cutoff.

Topping up. $\mu^*_{\mathrm{TU},i} = (m_i - \alpha)_+$. Iron $J_{\mu^*_{\mathrm{TU},i}}$ on the served prefix, with $q \geq q^{\mathrm{LF}}$.

Without topping up. $K_i = \mu^*_i + \alpha - m_i \geq 0$. The regular coefficient is $h_i(\theta) = J_i(\theta) + K_i/[\alpha f(\theta)] = \theta + [\mu^*_i - G_i(\theta)]/[\alpha f(\theta)]$ on $[\underline{\theta}, \kappa_i]$.

Include atom mass $\mu^*_i \underline{\theta}/\alpha$, iron on $[\underline{\theta}, \kappa_i]$, and leave higher types at laissez-faire.

Higher weights. $A(\theta) = \int_{\underline{\theta}}^{\theta} \Delta\omega\, \mathrm{d}F$ is nondecreasing, so $\Delta\mu^* - A(\theta)$ crosses zero at most once from above. Also $\Delta K = \Delta\mu^* - \Delta m \leq 0$.

A spread. $A \geq 0$ and $A(\underline{\theta}) = A(\overline{\theta}) = 0$. The theorem gives $\Delta\mu^* \geq 0$, while $\Delta K = \Delta\mu^*$. The sign pattern of Δh can be more complex.

◂ Higher weights: TU ◂ Higher weights: NTU

◂ Spread: TU ◂ Spread: NTU ▸ Numerical example

Negative correlation: numerical figures

$\theta \sim U[0.1, 10.1]$, $v(q) = \log q$, $c = 1$, $\alpha = 0.57$ and $\omega(\theta) = d + Be^{-k(\theta-0.1)}$.

	Baseline	Higher weights	Spread
d	0.03	0.13	0.03
B	3	3	4.4286
k	0.4	0.4	0.6
$\mathbf{E}[\omega]$	0.7663	0.8663	0.7663
μ^*_{TU}	0.1963	0.2963	0.1963
μ^*	0.2625	0.2977	0.3320
K	0.0663	0.0014	0.1358
Upper cutoff, TU	1.7946	2.7803	1.2927
Upper cutoff, NTU	8.6783	10.0625	7.4539
Free quantity, NTU	2.1187	2.3692	1.9224

The spread uses $B = 4.5(1 - e^{-4})/(1 - e^{-6})$, preserving the mean.

Back to identities
Back: higher weights
Back: spread
Main: higher weights
Main: spread